MATH 425a Homework 5

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Problem 1: 2.4

Write out a proof that the discrete metric on a set M is actually a metric.

Proof

Clearly, if the discrete metric d is defined by

$$d(x,y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

then clearly d is positive definite and symmetric: $d(x,y) \geqslant 0$, $d(x,y) = 0 \iff x = y$, and $d(x,y) = d(y,x)$. All that remains to be shown explicitly is that d also satisfies triangle inequality. The table below shows all possibilities, and indeed d satisfies triangle inequality.

<table>
<tr><th>$d(x,y)$</th><th>$d(y,z)$</th><th>$x = z$?</th><th>$d(x,z)$</th><th>$d(x,z) \leqslant d(x,y) + d(y,z)$?</th></tr>
<tr><td>0</td><td>0</td><td>Yes</td><td>0</td><td>Yes</td></tr>
<tr><td>0</td><td>1</td><td>No</td><td>1</td><td>Yes</td></tr>
<tr><td>1</td><td>0</td><td>No</td><td>1</td><td>Yes</td></tr>
<tr><td rowspan="2">1</td><td rowspan="2">1</td><td>No</td><td>1</td><td>Yes</td></tr>
<tr><td>Yes</td><td>0</td><td>Yes</td></tr>
</table>

Case-by-case verification of triangle inequality

□

Problem 2: 2.6

For $p, q \in [0, \pi/2)$ let

$$d_s(p, q) = \sin|p - q|.$$

Determine whether d_s is a metric.

Solution

Since, for $x \in [0, \pi/2)$, we always have $\sin x \geqslant 0$ and $\sin x = 0$ if and only if $x = 0$, d_s is positive definite. It's also obvious that $d_s(p, q) = \sin|p - q| = \sin|q - p| = d_s(q, p)$. Hence d_s is also symmetric. All that remains to verify is the triangle inequality. First note that, if $x, y \in [0, \pi/2)$, then $x - y \in (-\pi/2, \pi/2)$, and on this interval we have $\sin x = \sin|x| = |\sin x|$. Then, applying the sum-to-product formula we have, for $a, b, c \in [0, \pi/2)$,

$$\begin{aligned}
\sin|a - c| &= |\sin(a - c)| \\
&= |\sin((a - b) + (b - c))| \\
&= |\sin(a - b)\cos(b - c) + \cos(a - b)\sin(b - c)| \\
&\leqslant |\sin(a - b)\cos(b - c)| + |\sin(b - c)\cos(a - b)| \\
&\leqslant |\sin(a - b)||\cos(b - c)| + |\sin(b - c)||\cos(a - b)| \\
&\leqslant |\sin(a - b)| + |\sin(b - c)| \\
&= \sin|a - b| + \sin|b - c|
\end{aligned}$$

The last step shows that the triangle inequality holds. Hence d_s is a metric.

Problem 3: 2.7

Prove that every convergent sequence (p_n) in a metric space M is bounded, i.e., that for some $r > 0$ and some $q \in M$, for all $n \in \mathbb{N}$ we have $p_n \in M_r q$.

Proof

Since (p_n) converges, suppose $(p_n) \to p$, the limit. We may pick any $\epsilon > 0$. Then, by the definition of convergence, there exists $N \in \mathbb{N}$ such that $m \geqslant N \implies d_M(p_m, p) < \epsilon$. In other words, all terms of the sequence starting from p_m are within $M_\epsilon p$, the ϵ-neighborhood of p.

On the other hand, since N is finite, the set $\{p_1, \ldots, p_{m-1}\}$ is finite. If we set

$$r = \max\{d_M(p_1, p), d_M(p_2, p), \ldots, d_M(p_{m-1}, p), \epsilon\} + 1$$

then $d_M(p_i, p) < r$ for all p_i's of the sequence (p_n). Hence (p_n) is bounded and with every term within $M_r p$. □

Problem 4: 2.9

A sequence $(x_n) \in \mathbb{R}$ *increases* if $n < m$ implies $x_n \leqslant x_m$. It **strictly increases** if $n < m$ implies $x_n < x_m$. It **decreases** or **strictly decreases** if $n < m$ implies $x_n \geqslant x_m$ or $x_n > x_m$, respectively. A sequence is **monotone** if it increases or it decreases. Prove that every sequence in $\mathbb{R}$ which is monotone and bounded converges in $\mathbb{R}$.

Proof

Since a monotone sequence is either increasing or decreasing, we will start by checking the first case. Let (a_n) be an increasing sequence that is bounded. Then, by the L.U.B. property, we may proceed and define

$$s = \sup(A) \text{ where } A = \{a_n \mid n \in \mathbb{N}\}.$$

By the result of the last problem from discussion on Sept.1, we know that, given $\epsilon > 0$, there exists $a_i \in A$ such that

$$s - \epsilon < s - \frac{\epsilon}{2} \leqslant a_i \leqslant s. \tag{1}$$

Then, since (a_n) is increasing, we also have

$$j > i \implies a_i \leqslant a_j \leqslant s. \tag{2}$$

Combining inequalities (1) and (2) above, we have

$$\text{for all } \epsilon > 0, \text{there exists } i \in \mathbb{N} \text{ such that } j > i \implies s - \epsilon < a_j \leqslant s,$$

which is nothing else but the statement that (a_n) converges to s.

The proof for a decreasing sequence is almost analogous, except we have to use infimum as opposed to supremum. □

Problem 5: 2.13

Assume that $f : M \to N$ is a function from one metric space to another which satisfies the following condition:

$$\boxed{\text{If a sequence } (p_n) \text{ in } M \text{ converges then the sequence } (f(p_n)) \text{ in } N \text{ converges.}}$$

Prove that f is continuous.

Proof

If we can show that $(p_n) \to p \in M$ implies $(f(p_n)) \to fp \in N$, then we are done if we apply the first half of the proof for Pugh's Theorem 4 on page 65 [1]. Now define (p'_n) by the sequence $(p, p_1, p, p_2, p, p_3, \dots)$ in M.

Clearly this sequence is still converging to p. By the convergence of (p_n), given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geqslant N \implies d_M(p_n, p) < \epsilon$. Since p_n corresponds to p'_{2n}, we have if $m \geqslant 2N$ then $d_M(p'_m, p) < \epsilon$. This inequality either comes from the convergence of (p_n) or from the fact that $d_M(p,p) = 0$, depending on whether m is even or odd, since there is an alternating pattern between terms from (p_n) or just p itself.

Now, by the assumption given in the problem, since (p'_n) converges in N we know $(f(p'_n))$ converges to something in M. Now look at $(fp, fp, fp, \dots)$, a subsequence of $(f(p'_n))$. Clearly this sequence converges to fp. By Pugh's Theorem 1 on page 60 [23], we know that fp must also be the limit of $(f(p'_n))$. Therefore $(f(p_n))$, another subsequence of $(f(p'_n))$, must also converge to this same limit fp.

Having shown that $(f(p_n)) \to fp$, we may now apply Pugh's Theorem 4 and claim that f is continuous. $\square$

Problem 6: 2.14

The simplest type of mapping from one metric space to another is an **isometry**. It is a bijection $f : M \to N$ that preserves distance in the sense that for all $p, q \in M$ we have

$$d_N(fp, fq) = d_M(p, q).$$

If there exists an isometry from M to N then M and N are said to be **isometric**, $M \equiv N$. Isometric metric spaces are indistinguishable as metric spaces.

(a) Prove that every isometry is continuous.

(b) Prove that every isometry is a homeomorphism.

(c) prove that $[0, 1]$ is not isometric to $[0, 2]$.

Solution

Proof: Part (a)

Given an isometry $f : M \to N$, if we set $\delta = \epsilon$, then

$$d_M(p,q) = d_N(fp, fq) \implies \text{if } d_M(p,q) < \delta \text{ then } d_N(fp,fq) < \epsilon,$$

which shows that f is continuous. □

Proof: Part (b)

From what is provided and proven, we already know f is bijective and continuous. All that remains to show is that f is bicontinuous, i.e., f^{-1} is also continuous.

Note that $f^{-1} : N \to M$ is defined by $f(x) \mapsto x$ and $f^{-1} \circ f = \mathrm{id}_M$. Therefore we have

$$d_N(fp, fq) = d_M(p,q) = d_M((f^{-1}fp, f^{-1}fq),$$

which implies f^{-1} is also an isometry. By part (a), f^{-1} is continuous and therefore f is a homeomorphism.

□

Proof: Part (c)

Suppose $[0,1] \equiv [0,2]$, then there exist $x, y \in [0,1]$ such that

$$\begin{cases} f(x) = 0 \\ f(y) = 2 \end{cases} \quad \text{and } |x - y| = |f(x) - f(y)| = 2.$$

This is absurd since

$$\sup_{x,y \in [0,1]} |x - y| = 1 < 2.$$

Therefore $[0,1]$ is not isometric to $[0,2]$. □