MATH 425a Problem Set 6

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Problem 1

Recall that we gave our definition of continuity for a function $f : M \to N$ between two metric spaces in terms of ϵ and δ. We also defined sequential continuity in class: f sends convergent sequences in M to convergent sequences in N, with limits being sent to limits. You proved in the last problem set that the part about limits being sent to limits is actually redundant. We proved in class that continuity is equivalent to sequential continuity. We also defined topological continuity in class: the preimage of any open subset of N is an open subset of M.

(1) Write out a careful proof that continuity is equivalent to topological continuity.

(2) Suppose that we were to swap "open" with "closed" in the definition of topological continuity. Prove that the resulting definition would be equivalent.

Solution

(1) We want to show that f is continuous if and only if the preimage of open sets are open sets.

For $\Longrightarrow$, suppose f is continuous. Further suppose $S \subset N$ is open and $f^{-1}(S) \subset M$ is not. By definition, there exists $p \in f^{-1}(S)$ such that

$$\boxed{\text{For all } r > 0 \text{ we can find some } p' \in M_r p \text{ but } p' \notin f^{-1}(S).}$$

On the other hand, since $p \in f^{-1}(S)$, we know $fp \in S$. By the openness of S, there exists $\epsilon > 0$ such that $M_\epsilon(fp) \subset S$.

Fix this ϵ. By the continuity of f, we can also find $\delta > 0$ such that

$$\boxed{\text{For } q \in f^{-1}(S), \text{ if } d_M(p,q) < \delta \text{ then } d_N(fp, fq) < \epsilon.}$$

Also fix this δ. By the assumption of $f^{-1}(S)$ being not open, there exists p_δ satisfying (I) $d_M(p, p_\delta) < \delta$, (II) $p_\delta \notin f^{-1}(S)$, and (III) $d_N(fp, f(p_\delta)) < \epsilon$. A contradiction immediately appears as (II) implies $f(p_\delta) \notin S$ while (III) implies the contrary. Therefore the assumption that $f^{-1}(S)$ is not open must be false. Hence the **open set condition** is satisfied.

Now we try to show $\Longleftarrow$. Suppose f meets the open set condition. Let $S \subset N$ be an open set and pick $f(s) \in S$ (as the image of some $s \in M$). Then by the openness of S there exists $\epsilon > 0$ satisfying $N_\epsilon(f(s)) \subset S$, and by the openness of $f^{-1}(S)$ there exists $\delta > 0$ satisfying $M_\delta s < \delta$. This is precisely what makes f continuous: given $s \in N$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_N(s, s') < \delta \implies d_M(fs, fs') < \epsilon.$$

Therefore open set condition $\implies$ continuity. □

(2) This is immediate from the fact that

$$(f^{-1}(S))^c = f^{-1}(S^c).$$

(Recall that we already know openness is dual to closedness, so if f is continuous and if S and $f^{-1}(S)$ are both open, then S^c and $(f^{-1}(S))^c$ are both closed.)

Problem 2

Let (M, d_M) and (N, d_N) be metric spaces.

(1) Suppose that d_M is the discrete metric. Describe all continuous functions from M to N.

(2) Now suppose (M, d_M) is connected and d_N is the discrete metric. Describe all continuous functions from M to N.

Solution

(1) Note that if d_M is a metric and if we set $\delta < 1$, say $\delta = 0.5$ for example, then this δ satisfies the $\epsilon - \delta$ definition of continuity for all $\epsilon > 0$. This is because, in a discrete metric, given $p \in M$, if $d_M(p, q) < 0.5$ then the only possibility is $q = p$. Then $d_N(fp, fq) = 0 < \epsilon$ for all $\epsilon > 0$. Therefore *all* functions from M to N are continuous.

(2) Claim: if M is connected and N is equipped with the discrete metric d_N, then $f : M \to N$ is continuous if and only if it is a constant function. Below I will give a proof of this claim.

The $\Longleftarrow$ direction is obvious: $d_N(fp, fq) = 0$ for all $p, q \in M$. Therefore, regardless of the values of ϵ and δ, the $\epsilon - \delta$ condition always holds.

For the $\Longrightarrow$ direction, we will first prove a lemma:

Lemma

The continuous image of a connected set is connected.

Proof

Suppose $f : M \to N$ is continuous and $S \subset M$ is connected. For contradiction, suppose $f(S) = T \in N$ is disconnected. Then there exist a proper clopen subset $T_1 \subsetneq T$. By the topological definition of continuity (open/closed set conditions), $f^{-1}(T_1) \subsetneq T$ must also be a clopen and proper subset of S ($T \setminus T_1$ being nonempty implies $f^{-1}(T) \setminus f^{-1}(T_1)$ being nonempty). However, since M is connected, it does not have a proper clopen subset. Contradiction. Therefore T must be connected. □

Coming back to the main proof, first of all, it is not hard to see that all subsets of the discrete metric space (N, d_N) are clopen. Pick an arbitrary $S \subset N$. It is closed because, if a sequence $(p_n) \in S$ converges in N, then all sufficiently late terms are all one single point which is already in the subset S. On the other hand, it is open because, if we pick $r = 1/2$ and pick an arbitrary $p \in s$, then $N_r p = \{p\}$ which is a subset of S. We will soon need the fact that singletons in a discrete metric space are clopen.

The next thing to notice is that singletons are the only connected subsets of a discrete metric space. Singletons are connected because they do not have proper clopen subsets[1]. On the other hand, each (sub)set containing more than one element has singleton proper clopen subsets and are therefore disconnected.

If we look at the problem now, since M is connected and f is continuous, it follows that $f(M)$ is also connected. Hence the only possibility is that $f(M)$ is a singleton, i.e., f is a constant function. □

Problem 2.18 (Pugh)

Is $\mathbb{R}$ homeomorphic to $\mathbb{Q}$? Explain.

[1] In Pugh's book, a proper subset of M is a subset of M that is "neither the empty set nor M" (Pugh 86).

Solution

No, $\mathbb{R}$ is not homeomorphic to $\mathbb{Q}$ — they don't even have the same cardinality, and no bijection can exist between them, not to mention the existence of bicontinuous function.

Problem 2.19 (Pugh)

Is $\mathbb{Q}$ homeomorphic to $\mathbb{N}$? Explain.

Solution

Again, no. Suppose $\mathbb{Q} \cong \mathbb{N}$, then there exists a bicontinuous bijection $f : \mathbb{Q} \to \mathbb{N}$. Suppose $f(x) = 1$. By the injectivity of f we know that $f(y) = 1$ only if $y = x$.

To derive a contradiction, consider $\epsilon = 1$. Then we want to find $\delta > 0$ satisfying

$$\boxed{\text{If } |y - x| < \delta \text{ then } |f(y) - f(x)| < \epsilon.}$$

We know that if $|f(y) - f(x)| < \epsilon = 1$ then $f(y) = f(x)$, since the "distance" between natural numbers are all integers. Yet, no matter how small we set δ to be, we can always find a rational y with $|y-x| < \delta$ and $y \neq x$. (Recall that we've proven in a previous problem set that, given $x \in \mathbb{Q}$, there's no smallest $y \in \mathbb{Q}$ satisfying $y > x$.) Therefore the $\epsilon - \delta$ condition cannot be met, and f is not continuous at x. We conclude that $\mathbb{Q}$ and $\mathbb{N}$ are not homeomorphic.

Problem 2.23 (Pugh)

Prove that $(0, 1)$ is an open subset of $\mathbb{R}$ but not $\mathbb{R}^2$ when we think of $\mathbb{R}$ as the x-axis in $\mathbb{R}^2$.

Solution

To show $(0, 1)$ is open in $\mathbb{R}$, pick arbitrary $p \in (0, 1)$. If we set $r = \max(p, 1 - p)$, then $\mathbb{R}_r(p) \subset (0, 1)$. A quick justification:

$$\begin{cases} 0 < x \leqslant 0.5 \implies \mathbb{R}_r(p) = (p-p, p+p) = (0, p) \subset (0, 1) \\ 0.5 < x < 1 \implies \mathbb{R}_r(p) = (2p-1, 1) \subset (0, 1) \end{cases}.$$

However, it is not open open in $\mathbb{R}^2$ because, if we pick $q \in (0,1)$, no matter how small $r > 0$ is, the point $q' = (q, r/2)$ is not on the x-axis and thus $q \notin (0,1)$. On the other hand, $d_{\mathbb{R}}(q, q') = r/2 < r$.

Problem 2.25 (Pugh)

Prove directly from the definition of closed set that every singleton subset of a metric space M is a closed subset of M. Why does this imply that every finite set of points is also a closed set?

Solution

If we were to build a sequence out of a singleton $\{p\}$, the only possibility is $(p, p, p \dots)$ which clearly converges to $p \in \{p\}$. Hence the singleton contains *all* its limit points and is therefore closed.

Now suppose we are give a finite set $S = \{s_1, s_2, \dots, s_n\}$. If we set

$$\epsilon < \inf_{1 \leqslant i,j \leqslant n} (d_M(s_i, s_j))$$

then it is shorter than the distance between *any* two (distinct) points in the set. Now suppose (s_n) is an arbitrary convergent sequence in S that converges to $s \in M$. Fix ϵ as constructed above. By the definition of convergence there exists $N \in \mathbb{N}$ such that $m \geqslant N \implies d_M(s_m, s) < \epsilon$. Since ϵ is already smaller than the distance between *any* two (distinct) points in the set, s_m and s cannot be distinct. Hence we are left with the conclusion that all s_m's after s_N are the same and they are precisely s, the limit point. Therefore $s \in S$ and this set is closed.