MATH 425a Problem Set 7

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Problem 1

Let (M, d) be a metric space and let $S \subset M$ be a subset.

(1) Recall that the closure of S in M, denoted by $\overline{S}$, is by definition the set of all limits of S in M. Prove that $\overline{S}$ is closed in M, that it contains S, and that it is the smallest closed subset of M which contains S.

(2) A point $p \in M$ is called an **interior point** if there is some $r > 0$ such that $B_r(p) \subset S$. The **interior** of S, denoted by $\text{int}(S)$, is by definition the set of all interior points of S. Prove that S is an open subset of M if and only if $S = \text{int}(S)$.

Solution

(1) This part contains three questions, and we will show them one by one, in the order that they are asked.

(I) Showing $\overline{S}$ is closed is equivalent to showing that $\overline{S}$ contains all its limits. Suppose $s \in M$ is a limit point of $\overline{S}$, then some sequence $(s_n) \in \overline{S}$ converges to s. Therefore, given $\epsilon/2 > 0$, there exists $N_s \in \mathbb{N}$ such that

$$m \geqslant N_s \implies d(s_m, s) < \frac{\epsilon}{2}.$$

Pick any m satisfying the inequality above. Since such s_m is in $\overline{S}$, we know that there exists a sequence $((p_m)_n)_{n \in \mathbb{N}} \in S$ that converges to s_m. Therefore, given the same $\epsilon/2 > 0$ there exists $N_p \in \mathbb{N}$ such that

$$k \geqslant N_p \implies d((p_m)_k, s_m) < \frac{\epsilon}{2}.$$

Then, by triangle inequality, we have

$$d((p_m)_k, s) \leqslant d((p_m)_k, s_m) + d(s_m, s) < \epsilon.$$

Hence we have shown that, for any $\epsilon > 0$, we can find a point in S, $(p_m)_k$, who's within the ϵ-neighborhood of s. Setting $\epsilon = 1, 1/2, 1/3, \dots$ gives as a sequence of $(p_m)_k$'s which converge to s. Therefore s is a limit point of not only $\overline{S}$ but also S. This means $s \in \overline{S}$, and hence $\overline{S}$ is closed.

(II) To show $S \subset \overline{S}$, pick any $s \in S$ and consider the sequence $(s, s, s \dots)$. Clearly this sequence converges to s which implies $\in \overline{S}$. Hence $S \subset \overline{S}$.

(III) To show $\overline{S}$ is the smallest closed subset of M containing S, consider any closed subset $T \subset M$ containing S. Let s be a limit point of S. Then there exists a sequence $(s_n) \in S$ that converges to s. Since $S \subset T$, it follows that $(s_n) \in T$. By the closedness of T we see $s \in T$. Because s is arbitrary, we conclude that T contains all limit points of S, i.e., $\overline{S} \subset T$. Hence $\overline{S}$ is the smallest closed subset of M containing S.

(2) For $\Longrightarrow$, if S is open, then for each $s \in S$ there exists $r > 0$ with $B_r(s) \subset S$. Therefore all $s \in S$ are interior points of S, i.e., $S \subset \text{int}(S)$. On the other hand, $\text{int}(S) \subset S$ holds by definition since $\text{int}(S)$ is a set of points of S satisfying certain conditions. Hence $S = \text{int}(S)$.

Now for $\Longleftarrow$, suppose $S = \text{int}(S)$. Then $S \subset \text{int}(S)$; in other words, each $s \in S$ is an interior point with some $B_r(s) \subset S$. Note that this is exactly the definition of openness of S. Hence S is open.

Problem 2: 2.28 (Pugh)

A map $f : M \to N$ is **open** if for each open set $U \subset M$, the image set $f(U)$ is open in N.

(1) If f is open, is it continuous?

(2) If f is a homeomorphism, is it open?

(3) If f is an open, continuous bijection, is it a homeomorphism?

(4) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous surjection, must it be open?

(5) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous, open surjection, must it be a homeomorphism?

(6) If $: S^1 \to S^1$ is a continuous, open surjection, must it be a homeomorphism?

Solution

(1) No. Consider $f : \mathbb{R} \to \mathbb{Z}$. Note that any subset of $\mathbb{Z}$ is clopen (we just need the openness here — setting $r = 0.5$ then for any $x \in \mathbb{Z}$, the r-neighborhood of x is simply $\{x\} \subset \mathbb{Z}$). Therefore any f with codomain

$\mathbb{Z}$ is always open. Consider the floor function

$$f : \mathbb{R} \to \mathbb{Z} \text{ defined by } x \mapsto \lfloor x \rfloor .$$

Look at $x = 1$ and pick $\epsilon = 1/2$. For any $\delta > 0$ we always have $\lfloor x - \delta \rfloor < 1 \implies \lfloor x - \delta \rfloor \leqslant 0$. Therefore no $\delta > 0$ meets the $\epsilon - \delta$ criterion for $\epsilon = 1/2$. This means $f(x)$ is not continuous at $x = 1$. More generally, $f(x)$ is not continuous at all integers.

(2) Yes. If f is a homeomorphism then f^{-1} is continuous. Hence if $U \in M$ is open then $(f^{-1})^{-1}(U) = f(U)$ is open.

(3) Yes. The openness of f implies that the image of any open $U \subset M$ under f is open; this also means the preimage of any $U \subset M$ under f^{-1} is open. Therefore f^{-1} meets the open set condition and is continuous. Now we know f is bicontinuous and bijective, so it is by definition a homeomorphism.

(4) No. Consider the piecewise function

$$f(x) = \begin{cases} x + \pi & \text{for } x < -\pi \\ \sin(x) & \text{for } x \in [-\pi, \pi] \\ x - \pi & \text{for } x > \pi \end{cases}$$

and let $U = (-2, 2)$. Then the image is $[-1, 1]$, a closed interval.

In fact, a piecewise function with one constant interval suffices, since then we can come up with an open U and a half-open half-closed $f(U)$.

(5) Yes. To show f is a homeomorphism, from (3) we see that it suffices to show f is injective. Suppose for contradiction that f is not injective. Then there exist $x, y \in \mathbb{R}$ with $f(x) = f(y)$. Consider $U = [x, y]$ and $f(U) \subset \mathbb{R}$. Define $a = \min_{t \in [x,y]} f(t)$ and $b = \max_{t \in [x,y]} f(t)$. Note that the values of t is from $[x, y]$ rather than (x, y) to avoid situations in which $f(t)$ converges as $t \to x$ or $t \to y$.

In the degenerate case where $a = b$ we see that $f(x)$ is a constant function on the interval (x, y). Then $f(U)$ is a singleton which is not open with respect to standard Euclidean metric. Contradiction.

Other than the degenerate case we have $a \neq b$, so it's impossible that $\{f^{-1}(a), f^{-1}(b)\} = \{x, y\}$. Therefore, either a or b (or both), has to have a preimage from $[a, b] \setminus \{a, b\} = (a, b)$, the *closed* interval. Therefore $f(U)$ will be at least half-closed — either of form $\times, a]$ (we don't know what the other bracket and endpoint are, but they don't matter) or of form $[b, \times$. This again contradicts f's being open.

Therefore f must be injective, and by (3) we see that it is a homeomorphism.

(6) No. Each point on the unit circle has form $(\cos\theta, \sin\theta)$, and S^1 can be defined as

$$\{(\cos\theta, \sin\theta) \mid \theta \in [0, 2\pi)\}.$$

If we define $f : S^1 \to S^1$ by

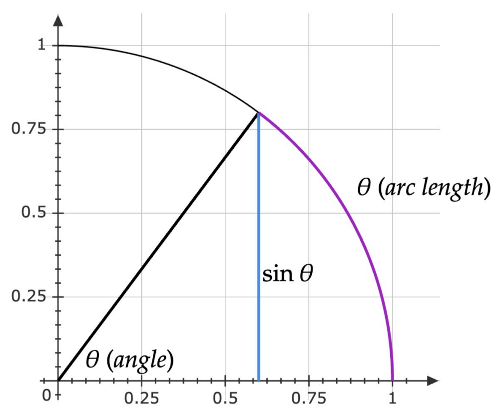
$$(\cos\theta, \sin\theta) \mapsto (\cos(2\theta), \sin(2\theta))$$

it follows that this function is continuous, open, and surjective†. Yet it is not a homeomorphism since it is not injective — $(\cos 0, \sin 0) = (1, 0) = (\cos(2\pi), \sin(2\pi))$ while $(\cos 0, \sin 0) \neq (\cos\pi, \sin\pi)$.

Proof of continuity, openness, and surjectivity of the last example above

Clearly if we define $\mathcal{S} = \{(\cos\theta, \sin\theta) \mid \theta \in [0, \pi)\}$ then $f(\mathcal{S})$ is already S^1. Hence f is surjective.

Now we will show that f is continuous using the $\epsilon - \delta$ definition. Pick $\epsilon > 0$ and a point $(\cos\theta, \sin\theta) \in S^1$ with $\theta \in [0, 2\pi)$. Note that $\theta > \sin\theta$ for all $\theta \in (0, 2\pi)$. (This can be justified using Taylor expansion, but an easier way is to think of a unit circle. θ is the arc length corresponding to a central angle θ with one side on the x-axis whereas $\sin\theta$ is the distance between the other endpoint and the x-axis. See figure below.



Furthermore, notice that, since $\sin x$ is concave down on the interval $(0, \pi/2)$, it takes the graph longer to double its value. Hence $\arcsin(2x) > 2\arcsin(x)^\dagger$.

†Since you asked for this:

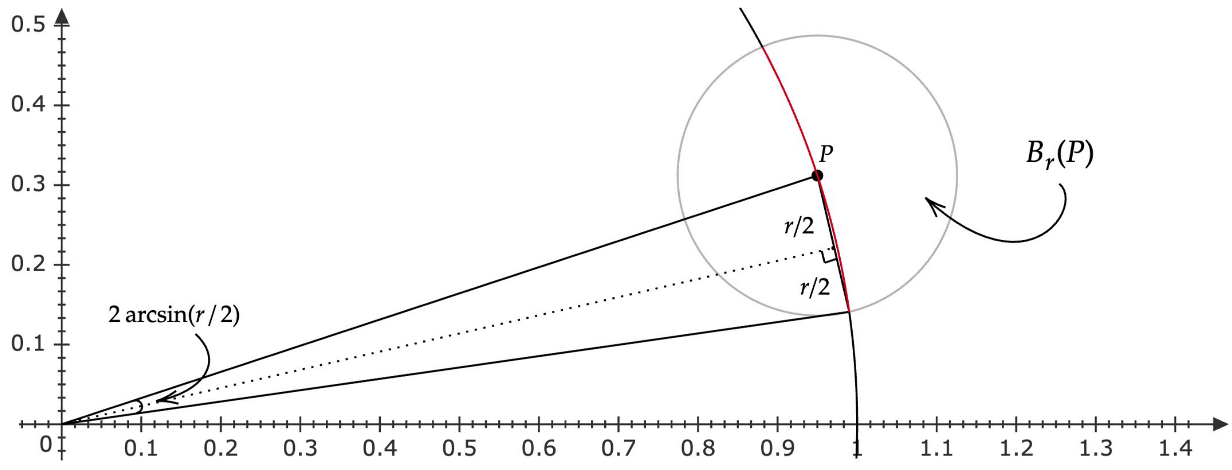
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}[\arcsin(2x) - 2\arcsin(x)] = \frac{2}{\sqrt{1-4x^2}} - \frac{2}{\sqrt{1-x^2}} > 0 \text{ for } x \in (0, \frac{\pi}{2}) \\ \arcsin(2 \cdot 0) = 2\arcsin(0) \end{cases}$$

The two equations above imply that $\arcsin(2x) > 2\arcsin(x)$ indeed holds for $x \in (0, \pi/2)$.

Now, pick any $p = (\cos\theta, \sin\theta) \in S^1$ the domain, and pick $\epsilon > 0$. We want to show that

$$\boxed{\text{there exists } \delta > 0 \text{ such that } d(p, q) < \delta \implies d(fp, fq) < \epsilon.}$$

First notice that, given $r > 0$, all the points on S^1 whose distance to $p < r$ all have form $(\cos\alpha, \sin\alpha)$ where $\alpha \in (\theta - 2\arcsin(r/2), \theta + 2\arcsin(r/2))$. Refer to the following diagram:



The two radii with a third side of length r form an isosceles triangle. The median from the apex to the base is perpendicular to the base, thus creating two right triangles, both with hypotenuse 1 and one cathetus $r/2$. Hence the apex angle is $2\arcsin(r/2)$. It is clear that any $\alpha \in (\theta - 2\arcsin(r/2), \theta + 2\arcsin(r/2))$ is enclosed within $B_r(p)$ and, with respect to S^1, the line segment marked in red. It's also clear that if α is not in this interval then $d(p,q) \geqslant r$.

Now, we will define $\epsilon' = \min(\epsilon, |\cos\theta|)$. Doing so ensures that $B_{\epsilon'}(p)$ never crosses the x-axis so we don't have to deal with the $0 = 2\pi$ issue for circles. If we set $\delta = \epsilon'/2$, we see that

$$d(p,q) < \delta \implies q = (\cos\alpha, \sin\alpha) \text{ for some } \alpha \in (\theta - 2\arcsin(\frac{\epsilon'}{4}), \theta + 2\arcsin(\frac{\epsilon'}{4})).$$

Previously, we've deduced that $\arcsin(\epsilon'/2) > 2\arcsin(\epsilon'/4)$, so if

$$\alpha \in (\theta - 2\arcsin(\frac{\epsilon'}{4}), \theta + 2\arcsin(\frac{\epsilon'}{4})) \text{ then } \alpha \in (\theta - \arcsin(\frac{\epsilon'}{2}), \theta + \arcsin(\frac{\epsilon'}{2})).$$

Recall that our function f maps $(\cos\theta, \sin\theta)$ to $(\cos(2\theta), \sin(2\theta))$. Therefore

$$fq = (\cos(2\alpha), \sin(2\alpha)) \text{ for some } 2\alpha \in (2\theta - 2\arcsin(\frac{\epsilon'}{2}), 2\theta + 2\arcsin(\frac{\epsilon'}{2})).$$

This is precisely the statement that $d(fp, fq) < \epsilon' \leqslant \epsilon$. Hence we've shown, with the help of parametrization by θ, that

$$\boxed{d(p,q) < \delta \implies |\theta - \alpha| < \text{ something } \implies |\theta - \alpha| < \text{ something larger } \implies d(fp, fq) < \epsilon.}$$

This concludes that f is continuous.

The openness of f follows immediately that if, for the domain, the r-neighborhood of $s \in S^1$ is a subset of S^1, then the $(2r)$-neighborhood of $f(s)$ is a subset of S^1, the codomain. Therefore the image of an open set is open.

Original Attempt

Given any $\epsilon > 0$ and pick any $(\cos\theta, \sin\theta)$, let $\delta = \epsilon/2$ and define $\Delta\theta = \dfrac{\epsilon}{2\sqrt{2}}$ (we'll see the reason soon). If $|\theta' - \theta| < \Delta\theta$, then

$$\begin{aligned} d((\cos\theta,\sin\theta),(\cos\theta',\sin\theta')) &= \sqrt{|\cos\theta - \cos\theta'|^2 + |\sin\theta - \sin\theta'|^2} \\ &= \sqrt{\left| -2\sin\left(\frac{\theta+\theta'}{2}\right)\sin\left(\frac{\theta-\theta'}{2}\right)\right|^2 + \left|2\cos\left(\frac{\theta+\theta'}{2}\right)\sin\left(\frac{\theta-\theta'}{2}\right)\right|^2}. \end{aligned}$$

Using the fact that $\sin\left(\frac{\theta+\theta'}{2}\right), \cos\left(\frac{\theta+\theta'}{2}\right) \leqslant 1$ and $\sin x < x$ for all $x \in (0, 2\pi)$, we have

$$\begin{aligned} d((\cos\theta,\sin\theta),(\cos\theta',\sin\theta')) &\leqslant \sqrt{\left| -2\sin\left(\frac{\Delta\theta}{2}\right)\right|^2 + \left|2\sin\left(\frac{\Delta\theta}{2}\right)\right|^2} \\ &< \sqrt{|(\Delta\theta)^2 + (\Delta\theta)^2|} \\ &= \sqrt{2}\Delta\theta \\ &= \frac{\epsilon}{2} = \delta. \end{aligned}$$

On the other hand,

$$\begin{aligned} d((\cos(2\theta),\sin(2\theta),(\cos(2\theta'),\sin(2\theta')) &= \sqrt{|\cos(2\theta) - \cos(2\theta')|^2 + |\sin(2\theta) - \sin(2\theta')|^2} \\ &= \sqrt{|-2\sin(\theta+\theta')\sin(\theta-\theta')|^2 + |2\cos(\theta+\theta')\sin(\theta-\theta')^2|} \\ &\leqslant \sqrt{|-2\sin(\theta-\theta')|^2 + |2\sin(\theta-\theta')|^2} \\ &< \sqrt{|-2\Delta\theta|^2 + |2\Delta\theta|^2} \\ &= 2\sqrt{2}\Delta\theta = \epsilon. \end{aligned}$$

Therefore, given any $\epsilon > 0$ and any $x \in S^1$ the domain, we can always find $\delta => 0$ such that if $d(x,y) < \delta, d(fx, fy) < \epsilon$. In particular, since each point on S^1 is determined by θ, we've found a sufficiently small $\Delta\theta$ that allows the δ to meet the $\epsilon - \delta$ condition. Hence f is continuous.

To see that f is open, consider an open subset $\mathcal{S} \in S^1$. By this assumption, if we pick a point $x = (\cos\theta_1, \sin\theta_1) \in \mathcal{S}$, we can find $r > 0$ satisfying $d(x,y) < r \implies y \in \mathcal{S}$. From the previous computation we see that if $|\theta_2 - \theta_1| < r/\sqrt{2}$ then $d(x, (\cos\theta_2, \sin\theta_2)) < r$. Therefore, $(\cos\theta_2, \sin\theta_2) \in \mathcal{S}$ for all $\theta_2 \in (\theta_1 - r/\sqrt{2}, \theta_1 + r/\sqrt{2})$.

Now let go of the previous notations and consider any $p = (\cos\alpha, \sin\alpha) \in f(\mathcal{S})$. Clearly $q = (\cos\alpha/2, \sin\alpha/2)$ is a preimage of p (not inverse image because f is not bijective). By what we've said in the last paragraph, there exists $\epsilon > 0$ such that, for all $s \in S^1$, if $d(q,s) < r$, then $s \in \mathcal{S}$. In particular, all such s's are of form $(\cos\beta/2, \sin\beta/2)$ where $\beta/2 \in ((\alpha-\epsilon)/2, (\alpha+\epsilon)/2)$. Therefore, all points of form $(\cos\beta, \sin\beta)$ where $\beta \in (\alpha - \epsilon, \alpha + \epsilon)$ are in $f(\mathcal{S})$. This means we've just found a neighborhood of an arbitrary point in $f(\mathcal{S})$ that lies entirely in $f(\mathcal{S})$. Hence $\mathcal{S}$ open implies $f(\mathcal{S})$ open, and f is indeed an open mapping.

An Even Easier Way from 10/7's Discussion

Suppose a sequence $(\cos(p_n), \sin(p_n)) \in S^1$ converges to $(\cos(p), \sin(p)) \in S^1$. Then the sequence must also converge component-wise, i.e., $\cos(p_n) \to \cos(p)$ and $\sin(p_n) \to \sin(p)$. The continuity of arcsin (and arccos) guarantees that $(p_n) \to p$. On the other hand, by the continuity of $\cos, \sin$, and the mapping $x \mapsto 2x$, we know that the composite mappings $x \mapsto \cos(2x)$ and $x \mapsto \sin(2x)$ are both continuous. Hence $\cos(2(p_n)) \to \cos(2p)$

and $\sin(2(p_n)) \to \sin(2p)$. Therefore $(\cos(2(p_n)), \sin(2(p_n))) \to (\cos(2p), \sin(2p))$, from which we see that f preserves sequential continuity.

□

Problem 3: 2.30 (Pugh)

Consider a two-point set $M = \{a, b\}$ whose topology consists of the two sets, M and the empty set. Why does this topology not arise from a metric on M?

Solution

In this topology, since $\{a\}, \{b\} \notin \mathcal{T}$ we know that these two sets are not open. Suppose this topology did arise from some metric space (M, d_M). If we define $d_M(a, b) = r$ and set $\epsilon = r/2$, we see that

$$\{x \in M \mid x \in B_\epsilon(a)\} = \{a\} \subset \{a\}$$

which suggests $\{a\}$ is open. Contradiction. Therefore $\mathcal{T}$ does not arise from any metric.

Problem 4: 2.34 (Pugh)

Use the Inheritance Principle to prove the following:

Corollary

Assume that N is a metric subspace of M and also is a closed subset of M. A set $L \subset N$ is closed in N if and only if it is closed in M. Similarly, if N is a metric subspace of M and also is an open subset of M then $U \subset N$ is open in N if and only if it is open in M.

Solution

~~(First we try to prove that $L \subset N$ is closed in N if and only if it is closed in M. The $\Longleftarrow$ direction is fairly obvious: if L is closed in M then by Inheritance Principle $L \cap N$ is closed in N, and this intersection is precisely L. For $\Longrightarrow$, first note that if L is closed in N, then it contains all its limit points with respect to N, i.e., $L = \overline{L}_N$. We want to show that L is also closed in M, i.e., $L = \overline{L}_M$. Hence it suffices to show $\overline{L}_N = \overline{L}_M$ and, as usual, we need to show mutual inclusion between these two sets. On one hand, any point $p \in \overline{L}_N$ is the limit of some sequence $(p_n) \in L$ that converges in N. Since $N \subset M$, the sequence clearly also converges in~~

~~M, so $p \in \overline{L}_M$. This shows $\overline{L}_N \subset \overline{L}_M$. On the other hand, any $p' \in \overline{L}_M$ is the limit of some sequence $(p_n) \in L$ that converges in M. Since $(p_n) \in L$ and $L \subset N$, we also know $(p_n) \in N$, and by the closedness of N, it must also converge in N. Hence $p' \in \overline{L}_M \implies p' \in \overline{L}_N$ which shows the other inclusion. Therefore $L \subset N$ is closed in N if and only if it is closed in M.)~~

(When writing the paragraph above, I didn't assume inheritance principle to hold for closed sets. Now that I know I may freely use it, the proof above is no longer necessary.)

First assume $N \subset M$ is closed. Note that we have the following propositions:

(1) Inheritance: $U \subset N$ is closed in N if and only if $U = N \cap V$ for some closed $V \subset M$.

(2) WTS: $U \subset N$ is closed in N if and only if U is closed in M

Therefore it suffices to show the following statement:

U is closed in M if and only if $U = N \cap V$ for some closed $V \subset M$.

The $\implies$ direction is immediate because, if U is closed, then setting $V = U$ gives $U = N \cap V = N \cap U$. For $\impliedby$, since both N and V are closed in M, so is their intersection. Hence U is closed in M.

If we replace "closed" with "open", we get $U \subset N$ is open in N if and only if U is open in M.

Problem 5: 2.38 (Pugh)

Let X, Y be metric spaces with metrics d_X, d_Y, and let $M = X \times Y$ be their Cartesian product. Prove that the three natural metrics $d_E, d_{\max}$, and d_{sum} are actually metrics.

Solution

It is very clear that all three metrics are symmetric and nonnegative, and they all satisfy $d(p,q) = 0$ if and only if $p = q$. Therefore all that remains is to show that all three metrics satisfy the triangle inequality.

(1) For Euclidean product metric:

$$\begin{aligned}
d_E((x_1,y_1),(x_2,y_2)) &= \sqrt{d_X(x_1,x_2)^2 + d_Y(y_1,y_2)^2} \\
&\leqslant \sqrt{[d_X(x_1,x_3)+d_X(x_3,x_2)]^2 + [d_Y(y_1,y_3)+d_Y(y_3,y_2)]^2} && \text{(TI of } d_X, d_Y\text{)} \\
&= \sqrt{\begin{aligned}&d_X(x_1,x_3)^2 + d_X(x_3,x_2)^2 + d_Y(y_1,y_3)^2 + d_Y(y_3,y_2)^2 \\ &+ 2[d_X(x_1,x_3)d_X(x_3,x_2)] + 2[d_Y(y_1,y_3)d_Y(y_3,y_2)]\end{aligned}} \\
&\leqslant \sqrt{\begin{aligned}&d_X(x_1,x_3)^2 + d_X(x_3,x_2)^2 + d_Y(y_1,y_3)^2 + d_Y(y_3,y_2)^2 \\ &+ 2\sqrt{d_X(x_1,x_3)^2 + d_Y(y_1,y_3)^2}\sqrt{d_X(x_3,x_2)^2 d_Y(y_3,y_2)^2}\end{aligned}} && \text{(Cauchy)} \\
&= \sqrt{\left(\sqrt{d_X(x_1,x_3)^2 + d_Y(y_1,y_3)^2} + \sqrt{d_X(x_3,x_2)^2 + d_Y(y_3,y_2)^2}\right)^2} \\
&= \sqrt{d_X(x_1,x_3)^2 + d_Y(y_1,y_3)^2} + \sqrt{d_X(x_3,x_2)^2 + d_Y(y_3,y_2)^2} \\
&= d_E((x_1,y_1),(x_3,y_3)) + d_E((x_3,y_3),(x_2,y_2)).
\end{aligned}$$

(2) For $d_{\max}$:

$$\begin{aligned}
d_{\max}((x_1,y_1),(x_2,y_2)) &= \max\{d_X(x_1,x_2), d_Y(y_1,y_2)\} \\
&\leqslant \max\{[d_X(x_1,x_3)+d_X(x_3,x_2)], [d_Y(y_1,y_3)+d_Y(y_3,y_2)]\} && \text{(TI of } d_X, d_Y\text{)} \\
&\leqslant \max\{d_X(x_1,x_3), d_Y(y_1,y_3)\} + \max\{d_X(x_3,x_2), d_Y(y_3,y_2)\} \\
&= d_{\max}((x_1,y_1),(x_3,y_3)) + d_{\max}((x_3,y_3),(x_2,y_2)).
\end{aligned}$$

(3) For d_{sum}:

$$\begin{aligned}
d_{\text{sum}}((x_1,y_1),(x_2,y_2)) &= d_X(x_1,x_2) + d_Y(y_1,y_2) \\
&\leqslant d_X(x_1,x_3) + d_X(x_3,x_2) + d_Y(y_1,y_3) + d_Y(y_3,y_2) && \text{(TI of } d_X, d_Y\text{)} \\
&= [d_X(x_1,x_3) + d_Y(y_1,y_3)] + [d_X(x_3,x_2) + d_Y(y_3,y_2)] \\
&= d_{\text{sum}}((x_1,y_1),(x_3,y_3)) + d_{\text{sum}}((x_3,y_3),(x_2,y_2)).
\end{aligned}$$

Therefore all three metrics are indeed metrics.

Problem 6: 2.39 (Pugh)

(1) Prove that every convergent sequence is bounded. That is, if (p_n) converges in the metric space M, prove that there is some neighborhood $M_r q$ containing the set $\{p_n \mid n \in \mathbb{N}\}$.

(2) Is the same true for a Cauchy sequence in an incomplete metric space?

Solution

(1) The first part is exactly the same as problem 3 in HW5. By the convergence of (p_n), suppose $(p_n) \to p \in M$ and if we pick any $\epsilon > 0$, then we can always find $N \in \mathbb{N}$ satisfying

$$n \geqslant N \implies d_M(p_n, p) < \epsilon.$$

Therefore there are only finitely many terms, the $(n-1)$ terms to be exact, whose distance to p is not guaranteed to be $< \epsilon$. If we set

$$r = \max\{d_M(p_1, p), d_M(p_2, p), \dots, d_M(p_{(n-1)}, p), \epsilon\} + 1$$

then all points in the sequence is enclosed in $M_r p$, and this finishes the proof that (p_n) is bounded.

(2) Yes, and the same logic still applies! Suppose (p_n) is Cauchy. Let us pick any $\epsilon > 0$. It follows that we can always find $N \in \mathbb{N}$ satisfying

$$m, n \geqslant N \implies d_M(p_m, p_n) < \epsilon.$$

If we fix either m or n to be exactly N, we see that all terms of (p_n), starting from p_N, is enclosed in $M_\epsilon(p_N)$. Again, there are only finitely terms — $(N-1)$ to be precise — whose distance to p_N is not guaranteed to be $< \epsilon$. If we set

$$r = \max\{d_M(p_1, p_N), d_M(p_2, p_N), \dots, d_M(p_{(N-1)}, p_N), \epsilon\} + 1$$

then the entire sequence (p_n) is enclosed in $M_r(p_N)$. Hence it is bounded.

Problem 7: 2.43 (Pugh)

Assume that the Cartesian product of two nonempty sets $A \subset M$ and $B \subset N$ is compact in $M \times N$. Prove that A and B are compact.

Proof

Pick any sequence $(a_n) \in A$ and any $b \in B$. Consider the sequence $(a_n, b) \in A \times B$. By the compactness of $A \times B$ we know that this sequence has a convergent subsequence, which we call (a_{n_k}, b_k). Note that b_k is constant. Since (a_{n_k}, b_k) converges, it converges component-wise. Thus (a_{n_k}) converges. Therefore any arbitrarily chosen $(a_n) \in A$ has a convergent subsequence and we conclude A is compact. The proof of the compactness of B is analogous. □