# MATH 425a Problem Set 8

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**Problem 2.37 (Pugh)**

Construct a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous only at points of $\mathbb{Z}$.

**Solution**

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(\pi x) & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Before we show $f$ is only continuous at points of $\mathbb{Z}$, note that $\sin(\pi x) = 0$ if and only if $x \in \mathbb{Z}$. Also, $\sin(\pi x)$ is a continuous function.

We first show that $f$ is indeed continuous at points of $\mathbb{Z}$. Suppose we had a sequence $(p_n) \in \mathbb{R}$ that converges to $p \in \mathbb{Z}$. By the continuity of $\sin(\pi x)$, a function defined on the entire $\mathbb{R}$, we know that, for any $z \in \mathbb{Z} \subset \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } q \in \mathbb{R} \text{ and } |p - q| < \delta \text{ then } |\sin(p\pi) - \sin(q\pi)| < \epsilon, \text{ i.e., } |\sin(q\pi)| < \epsilon.$$

It's not hard to see that $|f(x)| \leqslant |\sin(\pi x)|$ for all $x \in \mathbb{R}$. Hence we now have a stronger statement: for any $z \in \mathbb{Z} \subset \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } q \in \mathbb{R} \text{ and } |z - q| < \delta \text{ then } |f(z) - f(q)| < \epsilon.$$

This is exactly the statement that $f(x)$ is continuous at all points of $\mathbb{Z}$.

Now, to show that $f$ is not continuous at points of $\mathbb{R} \setminus \mathbb{Z}$, consider $k \in \mathbb{R} \setminus \mathbb{Z}$. We know that $\sin(k\pi) \neq 0$. Since (I hope we can take this for granted) $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$, we are able to construct a sequence $(k_n) \to k$ that contains infinitely many rational terms and also infinitely many irrational terms. Then we have two subsequences, $(k_n)'$ and $(k_n)''$, of all the rational terms and irrational terms of $(k_n)$, respectively. Therefore the sequence $(f(k_n))$ has two subsequences $(f(k_n)')$ and $(f(k_n)'')$. Since $\sin(\pi x)$ is continuous, it preserves sequential convergence. Since $(k_n)'$ converges to $k$ just like its mother sequence, we know that the sequence $(\sin((k_n)'\pi))$ converges to $\sin(k\pi)$, and this sequence is precisely $(f(k_n)')$. (Notice that since we haven't specified whether $k$ is rational, we cannot say $(f(k_n)') \to f(k)$; however, it is safe to say that it converges to $\sin(k\pi)$, a function defined on entire $\mathbb{R}$.) On the other hand, $(f(k_n)'')$ is the constant sequence $(0, 0, \dots)$ which clearly converges to 0. If $f$ were continuous at $k$, then it preserves sequential convergence and hence $(f(k_n))$ converges. However $(f(k_n))$ has two subsequences that converge to different limits, and we are forced to the absurd conclusion that $(f(k_n))$ also converges to two different limits. Therefore $f$ is not continuous at $k$, and this concludes the proof that $f$ is continuous at $x$ if and only if $x \in \mathbb{Z}$.

**Remark**

Also consider the following $g : \mathbb{R} \to \mathbb{R}$, another function that is only continuous at points of $\mathbb{Z}$:

$$g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ \lfloor x + 0.5 \rfloor & \text{if } x \notin \mathbb{Q} \end{cases}$$

**Problem 2.47(a) (Pugh)**

Suppose $A, B \subset \mathbb{R}^2$. If $A \cong B$, are their complements homeomorphic?

**Solution**

Not necessarily. Consider $A = \mathbb{R}^2$ and $B = \mathbb{R} \times (-\pi/2, \pi/2)$. Then the function $f : A \to B$ defined by $(x, y) \mapsto (x, \arctan y)$ is a homeomorphism[1]. However, $A^c = \varnothing$ and $B^c$ is nonempty; they have different cardinalities. There cannot exist a homeomorphism between these two complements, so $A^c \ncong B^c$.

**Problem 3**

Let $(M, d)$ be a metric space, and $S \subset M$ a connected subset. Is the interior of $S$ connected? Prove or disprove.

[1] I posted a question here, asking for help to check if it is necessary to explicitly show that this is indeed a homeomorphism. If you believe it is necessary, the complete proof is in the comment section.

**Solution**

Not necessarily connected. Consider the union of two closed unit disks:

$$D = D_1 \cup D_2 \leqslant \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid (x-2)^2 + y^2 \leqslant 1\}.$$

Its interior of $D$ would be

$$\text{int}(D) = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid (x-2)^2 + y^2 < 1\} = \text{int}(D_1) \cup \text{int}(D_2).$$

This is because all the points in $D$ that are not interior points are on the two unit circles, i.e., the boundary of $D$:

$$\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid (x-2)^2 + y^2 = 1\}.$$

If we remove them, then we get $\text{int}(D)$, the (disjoint) union of two open disks, again with radius 1. This set is disconnected because $\text{int}(D_1), \text{int}(D_2)$ are open, but taking the complement suggests that they are also closed. Hence they are proper clopen subsets of $\text{int}(D)$, which makes $\text{int}(D)$ disconnected.

**Problem 4**

(1) Find a bounded set of real numbers with exactly 3 cluster points.

(2) Find a compact set of real numbers whose set of cluster points is infinite.

**Solution**

(1) Consider the set $S = S_1 \cup S_2 \cup S_3$ where $S_1, S_2, S_3$ are defined by

$$S_1 = \{1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, \dots\} = \{1 + \frac{1}{n+1} \mid n \in \mathbb{N}\}$$

$$S_2 = \{2 + \frac{1}{2}, 2 + \frac{1}{3}, 2 + \frac{1}{4}, \dots\} = \{2 + \frac{1}{n+1} \mid n \in \mathbb{N}\}$$

$$S_3 = \{3 + \frac{1}{2}, 3 + \frac{1}{3}, 3 + \frac{1}{4}, \dots\} = \{3 + \frac{1}{n+1} \mid n \in \mathbb{N}\}.$$

Obviously $S$ is bounded and there exist sequences in $S$ that converge to $1, 2$, and $3$, respectively. By Theorem 52(i) we know that $1, 2$, and $3$ are cluster points. Now it remains to show any other points in $S$ is not a cluster point. Let $s \in S$ be any number but an integer. By the construction it can only belong to one among $S_1, S_2, S_3$, so it must have the form $i + 1/j$ for some $i \in \{1, 2, 3\}$. Then the closest element to $s$ is $i+1/(j+1)$ and $i+1/(j-1)$. The distances are $1/j(j+1)$ and $1/j(j-1)$, respectively. If we set $\epsilon$ to be smaller than both of them then no point in $S$ is within the $\epsilon$-neighborhood of $s$, and hence $s$ cannot be a cluster point. Therefore $S$ has precisely 3 cluster points, $1, 2$, and 3.

(2) Consider the set

$$S = \left\{ \frac{1}{2^m} \middle| m \in \mathbb{N} \right\} \cup \left\{ \frac{1}{2^m}\left(1 + \frac{1}{n+1}\right) \middle| m, n \in \mathbb{N} \right\}$$

which is very similar to the example in (1), except now $S$ is the union of countably infinite such sets. By the same reasoning, $s \in S$ is a clustering point if and only if it is of form $1/2^m$, and there are countably infinite such points. Note that

$$\sup(S) = \frac{1}{2} \cdot (1 + \frac{1}{2}) = \frac{3}{4}$$

and

$$\inf(S) \geqslant 0 \text{ since } \frac{1}{2^m} > 0 \text{ and } (1 + \frac{1}{n+1}) > 0.$$

Hence $S$ is bounded by $[0, 3/4]$. Note that, in $S$, besides the "boring" constant convergent sequences, the only convergent sequences are those that converge to one the clustering points, since all other points are at least certain distance away from each other and it is impossible for them to be the limits of a convergent sequence. Hence $S$ contains all its limits and is closed. Therefore, by Heine-Borel Theorem, $S$ is compact, and this finishes the problem.

**Problem 5**

Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(x, y) \mapsto (\sin(xy^2), 3x^3y + xy^2)$$

is continuous.

**Solution**

Before proving that $f$ is continuous, we first look at a simpler case $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$(x, y) \mapsto xy.$$

Pick any $\epsilon > 0$ and $(x, y) \in \mathbb{R}^2$ and assume $d((x, y), (x', y')) < 1$ (we can always do so by using a min() function when defining $\delta$ later on). Then it follows immediately that both $|x - x'|$ and $|y - y'|$ are less than 1.

Since

$$\begin{aligned}
|xy - x'y'| &= |xy - (x' - x + x)(y' - y + y)| \\
&= |-(x' - x)(y' - y) - (x' - x)y - x(y' - y)| \\
&= |(x - x')(y - y') + x(y - y') + y(x - x')| \\
&\leqslant |(x - x')(y - y')| + |x(y - y')| + |y(x - x')| && \text{(Triangle inequality)} \\
&\leqslant |x - x'||y - y'| + |x||y - y'| + |y||x - x'| \\
&< \underbrace{\sqrt{|x - x'|^2 + |y - y'|^2}}_{d} + |x||y - y'| + |y||x - x'| && \text{(Cauchy-Schwarz)} \\
&< d + |y|d + |x|d && (\text{Since } d^2 > |x - x'|^2, |y - y'|^2) \\
&= d(1 + |x| + |y|),
\end{aligned}$$

if we define

$$\delta := \min\left(1, \frac{\epsilon}{1 + |x| + |y|}\right) > 0$$

then

$$d((x,y),(x',y')) < \delta \implies d(xy, x'y') < (1 + |x| + |y|)\delta \leqslant \epsilon.$$

Therefore the function $f(x, y) = xy$ from $\mathbb{R}^2$ to $\mathbb{R}$ is continuous.

Likewise, the functions $\mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto xy^2$ is continuous because it is simply the composite function $s(t, y) : \mathbb{R}^2 \to \mathbb{R}$ defined by $(t, y) \mapsto ty$ whereas $t(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is defined by $(x, y) \mapsto xy$. By the same token, the functions $\mathbb{R} \to \mathbb{R}$ defined by $(x, y) \mapsto 3x^3y$ and $(x, y) \mapsto xy^2$ are also continuous.

Notice that $\sin(xy^2)$ is the composite of two continuous functions (sin and the one we've shown above), so it is continuous. On the other hand, $(x, y) \mapsto 3x^3y + xy^2$ from $\mathbb{R}^2$ to $\mathbb{R}$ is also continuous because it is the *sum* of two continuous functions: given $\epsilon > 0$ we can pick two $\delta$'s for the two functions that correspond to $\epsilon/2$, and then picking the smaller one between the two $\delta$'s ensures that the *sum* function is also continuous.

Finally, if we look at $f : \mathbb{R} \to \mathbb{R}$ defined by $(x, y) \mapsto (\sin(xy^2), 3x^3y + xy^2)$, we know that both components are continuous. Therefore $f$ is continuous. (This can once again be easily shown using sequential continuity, but Prof. Siegel said what I've done above suffices.) □

**Problem 6**

Does a continuous function between two metric spaces send closed subsets to closed subsets? Prove or disprove.

**Solution**

No. For simplicity consider $f : \mathbb{R} \to \mathbb{R}$. We know that continuous functions map compact sets to compact — and thus closed — sets, so in order to come up with a counterexample for this problem we can make the domain unbounded so it is not compact. **Consider the exponential function $f(x) = e^x$ for $x \in \mathbb{R}$.** The domain is $\mathbb{R}$, clopen, but the range is $(0, \mathbb{R})$, not closed since 0 is a limit point outside the interval.

**Problem 7**

Give an example of an open cover of $(0,1)$ which has no finite subcover. Conclude that $(0,1)$ is not compact.

**Solution**

Similar to the example provided in the textbook, consider the covering

$$\mathcal{U} = \left\{ (\frac{1}{n}, \frac{n-1}{n}) \mid n \geqslant 2 \right\}.$$

For any finite covering $\mathcal{U}'$, suppose the scrap with the largest index has index $m$, then

$$\mathcal{U}' \subset \bigcup_{i=2}^{m} (\frac{1}{i}, \frac{i-1}{i})$$

which fails to cover $(0, 1/i)$ and $((i-1)/i, 1)$. Therefore $\mathcal{U}$ does not have a finite subcovering and $(0,1)$ is not covering compact.

**Problem 8**

Let $S \subset \mathbb{R}^2$ be the "closed topologist's sine curve", defined by

$$S := \{(x, \sin(1/x) \in \mathbb{R}^2 \mid x \in (0,1]\} \cup \{0\} \times [-1, 1].$$

Prove that $S$ is connected but not path connected.

**Solution**

For notations, denote the topologist's sine curve by $S^+$ and $\{0\} \times [-1,1]$ by $S_0$. It is obvious that $S^+$ is connected since it is a continuous image of a connected set $(0,1]$. Recall Pugh's Theorem 49 states if $S^+$ is connected, so is its closure. We will now show that $\overline{S^+} = S$.

**Lemma**

The closure of the (original) topologist's sine curve is the union of the curve with $\{0\} \times [-1,1]$.

*Proof.* Using the previous notation, we first show that $S^+ \cup S_0 \subset \overline{S^+}$. Pick any $s \in S^+ \cup S_0$. Either $s \in A^+$, i.e., $s$ is on the topologist's sine curve, or it is of form $(0, y_0)$ for some $y_0 \in [-1,1]$. Clearly, for the former case, the sequence $(s, s, \dots)$ converges to $s$. For the latter, first notice that $y_0 = \sin\theta$ for some $\theta \in [\pi, 3\pi]$ (we are not picking the interval $[0, 2\pi]$ as we usually would because 0 is not in the domain of the topologist's sine function). Therefore we know that $(1/\theta, \sin\theta) \in S^+$. Recall that sin is periodic with period $2\pi$, so all points of form $(1/(\theta + 2k\pi), \sin(\theta + 2k\pi))$ are also in $S^+$, and these points all have $y$-coordinate $y_0$. If we take $k = 1, 2, \dots$ then we have constructed a sequence that converges to $(0, y_0)$. This shows $s \in \overline{S^+}$ as well. Hence $S^+ \cup S_0 \subset S$.

For the other direction, we want to show that $\overline{S^+} \subset S^+ \cup S_0$. Pick any $p = (x_1, y_1) \in \overline{S^+}$ and we know that some sequence of points $(p_n) = ((x_n), (y_n))$ in $S^+$ converges to $p \in \mathbb{R}^2$. First notice two things:

(1) $x_1$ is nonnegative because each term of $(x_n)$ is positive, and

(2) $|y_1| \leqslant 1$ since each term of $(y_n)$ is between $[-1,1]$.

If $x_1 = 0$ then we know $p$ is on the $y$-axis. By the observations above we know $p \in S_0$. From the first part of the proof we know there actually exists a sequence converging to $p$.

Now we are left with the case $x_1 \neq 0$. Since $x \mapsto \sin(1/x)$ for $x \in (0,1]$ is a composite of two continuous functions, we know that this mapping is also continuous. Hence if $(x_n) \to x_1$ then

$$(y_n) = (\sin(x_n^{-1})) \to \sin(1/x_1) \text{ which we call } y_1.$$

Therefore $((x_n), (y_n)) \to (x_1, y_1) \in S^+$, and we've shown that $\overline{S^+} \subset S^+ \cup S_0$ and also $\overline{S^+} = S^+ \cup S_0$. □

Now, it follows that, since $S^+$ is connected, so is its closure $\overline{S^+}$ which is exactly the one given by the problem.

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To show that $S$ is not path connected, it suffices to show that no path exists between $(0,0)$ and any point

$(x, y) \in S^+$. Suppose there existed a continuous function $\gamma : [0,1] \to \mathbb{R}^2$ with $\gamma(0) = (0,0)$ and $\gamma(1) = (x, y)$. Since $\gamma$ is continuous, if we set $\epsilon = 1/2$ then there exists a $\delta > 0$ such that

$$\text{if } t < \delta \text{ then } d(\gamma(t), \gamma(0)) < \epsilon = \frac{1}{2}.$$

To visualize this statement, refer to the following diagram:

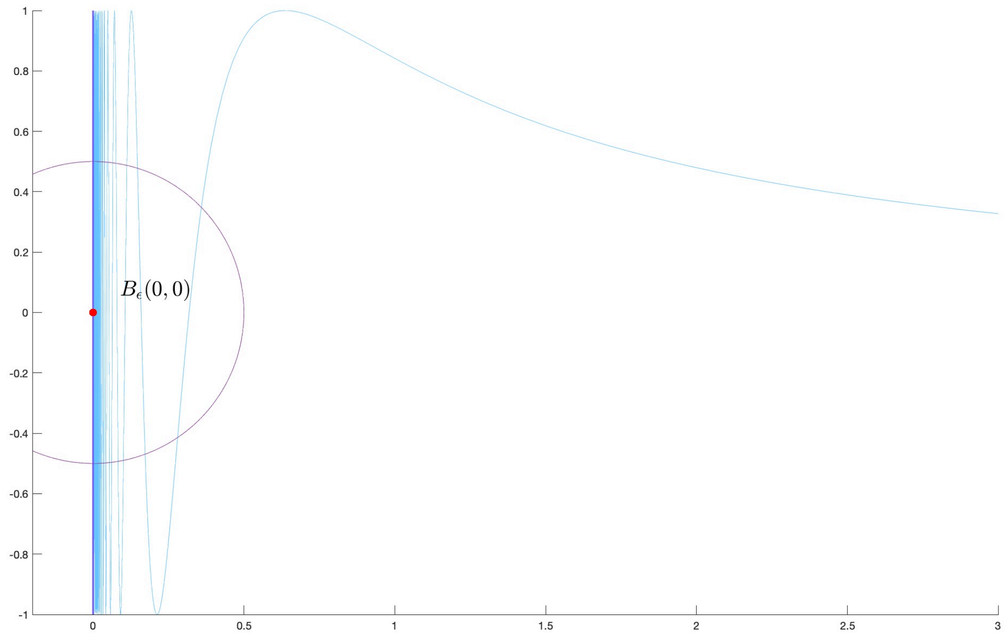


Figure 1: The 1/2 neighborhood of origin and the topologist's sine curve on $(0,3] \times [-1,1]$.

The intuition here is that, since the topologist's sine curve keeps jumping in and out of the disk $B_\epsilon(0,0)$, it is never possible to find a $\delta$.

Back to the proof — suppose $\gamma$ were continuous, then there exists a $\delta > 0$ corresponding to the $\epsilon = 1/2$. Call the point $\gamma(\delta) = (x^*, y^*)$. Now we narrow our focus down to $[0, \delta]$. Since $\gamma$ restricted to this domain is still continuous, and since the mapping $\mathbb{R}^2 \to \mathbb{R}$ defined by $(x, \sin(1/x)) \mapsto x$ is continuous, their composite $\psi : [0, \delta] \to \mathbb{R}$ is also continuous. Recall that continuous functions map connected sets to connected sets, so the image $\psi([0, \delta])$ must also be connected. On one hand, $\psi(0) = 0$; on the other hand, $\psi(\delta) = x^*$. Therefore the entire interval $[0, x^*]$ must be contained in the image of $\psi$. Hence for **any** $x' \in [0, x^*]$, there exists $t \in [0, \delta]$ satisfying $\psi(t) = x'$.

Now we proceed to construct a contradiction. Recall that $\sin((k + 0.5)\pi) = 1^2$ for $k \in \mathbb{Z}$. Given $x^*$ above, we can find a $k' \in \mathbb{Z}$ large enough such that

$$\frac{1}{(k' + 0.5)\pi} = x' < x^* \text{ while } \sin(1/x') = \sin((k' + 0.5)\pi) = 1.$$

This shows that, even though $x'$ is close enough to 0, $d(\gamma(x'), \gamma(0)) = d((x', 1), (0, 0)) > 1 > \epsilon$. Therefore our assumption of $\gamma$ being continuous must be false, and no path exists between $(0, 0)$ and **any** point on $S^+$. Therefore $S$ is not path-connected. $\square$

**Remark**

For a stronger argument, pick any $s \in S_0 = \{0\} \times [-1, 1]$ is at least 1. If its coordinate is $(0, y)$ then if we set $\epsilon < 1 - y$ we can actually use the same $\epsilon - \delta$ argument to show that there does not exist **any** path between **any** point on the curve and **any** point in $S_0$. This again shows that $S$ is not path-connected.

[2] I originally wrote something similar but wrong; credits to Yizhen Wu.