MATH 425a Problem Set #9

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Problem 1: 2.59 (Pugh)

Prove that every countable metric space (not empty and not a singleton) is disconnected. [Astonishingly, there exists a countable topological space which is connected. Its topology does not arise from a metric.]

Solution

Suppose, for contradiction, that we had a connected, countable metric space (M, d) with at least two points a and b. First thing to note is that, if there existed an d_0 with $0 < d_0 =< d_M(a, b)$ satisfying

$$d_M(a, x) \neq d_0 \text{ for all } x \in M,$$

then there exists a separation of M defined by

$$M = \{x \in M \mid d_M(x, a) < d_0\} \sqcup \{x \in M \mid d_M(x, a) > d_0\}$$

which contradicts M's being connected. Therefore such d_0 cannot exist; in other words, for all $\tilde{d} \in (0, d_M(a, b))$, there exists some $\tilde{x} \in M$ satisfying $d_M(a, \tilde{x}) = \tilde{d}$. Therefore the function $f : M \to (0, d_M(a, b))$ defined by

$$x \mapsto d_M(a, x)$$

is a surjection, and this suggests that M is uncountable. Contradiction. Therefore M cannot be connected and countable at the same time.

Problem 2: 2.71 (Pugh)

Let M and N be nonempty metric spaces.

(1) If M and N are connected, prove that $M \times N$ is connected.

(2) What about the converse?

(3) Answer the questions again for path-connectedness.

Solution

(1) We prove by taking the contrapositive. Suppose $M \times N$ were disconnected, then there exists a separation $A \sqcup B$. Note that, for all $m \in M$, we have $N \cong \{m\} \times N$ by $f : \tilde{n} \mapsto (m, \tilde{n})$ for $\tilde{n} \in N$, a homeomorphism. Since connectedness is preserved under homeomorphisms, it follows that, for all $m_0 \in M$ and $n_0 \in N$, $(\{m_0\} \times N) \cup (M \times \{n_0\})$ is also connected since it is the union of two "strips" whose intersection is (m_0, n_0). Then, fixing m_0 and consider

$$(\{m_0\} \times n) \cup \bigcup_{\tilde{n} \in N} (M \times \{\tilde{n}\})$$

which is $M \times N = P$ but also connected since each one of them is connected and that they all contain $\{m_0\} \times N$. Therefore P is connected.

(2) Suppose $M \times N$ is connected, then the continuous projection $(m, n) \mapsto m$ implies that M is a continuous image of a connected set, so it must be connected. Likewise for N.

(3) For $\implies$, pick any $(m_1, n_1), (m_2, n_2) \in M \times N$. Since M and N are assumed to be path-connected, there exist continuous $\gamma_1 : [0, 1] \to M$ and $\gamma_2 : [0, 1] \to N$ satisfying

$$\begin{cases} \gamma_1(0) = m_1 \\ \gamma_1(1) = m_2 \end{cases} \quad \text{and} \quad \begin{cases} \gamma_2(0) = n_1 \\ \gamma_2(1) = n_2 \end{cases} .$$

Now consider the function $\psi : [0, 1] \to M \times N$ defined by $t \mapsto (\gamma_1(t), \gamma_2(t))$. This is clearly continuous as both components are continuous, and it also satisfies $\psi(0) = (m_1, n_1)$ and $\psi(1) = (m_2, n_2)$. Therefore $M \times N$ is path-connected.

The steps for $\impliedby$ is analogous to that for connectedness, since homeomorphisms not only preserves connectedness but also path-connectedness.

Problem 3: 2.85 (Pugh)

Suppose that M is compact and that $\mathcal{U}$ is an open covering of M which is *redundant* in the sense that each $p \in M$ is contained in at least two members of $\mathcal{U}$. Show that $\mathcal{U}$ reduces to a finite subcovering with the same property.

Solution

If $\mathcal{U}$ is finite then we are immediately done. If not, by the compactness of M we know $\mathcal{U}$ has a finite (open) covering $\mathcal{U}' = \bigcup_{i=1}^{n} U_i$. Consider this subcovering. Notice that the subset $\mathcal{S} \subset M$ in which each point is covered by only one scrap is defined as

$$\mathcal{S} = M \setminus \bigcup_{1 \leqslant i < j \leqslant n} (U_i \cap U_j).$$

Since the intersection of two open sets are open and the intersection of (finitely many) open sets is open, taking the complement suggests S is a closed subset of the compact set M. Hence $\mathcal{S}$ itself is also compact.

Now consider the scraps from $\mathcal{U} \setminus \mathcal{U}'$. We know that, originally, using $\mathcal{U}$, every point in $\mathcal{S}$ is covered at least twice; now with $\mathcal{U}'$ they are only covered once. This means *every point in* $\mathcal{S}$ belongs to some scrap in $\mathcal{U} \setminus \mathcal{U}'$. Consider the collection of such craps $\mathcal{U}_{\mathcal{S}}$. By the compactness of $\mathcal{S}$, we know $\mathcal{U}_{\mathcal{S}}$ also has a finite subcovering which we call $\mathcal{U}_{\mathcal{S}}^*$. Then the union $\mathcal{U}' \cup \mathcal{U}_{\mathcal{S}}^*$ covers all $p \in M$ at least twice and, more importantly, it is indeed finite.

Problem 4: 2.97 (Pugh)

Is the set of dyadic rationals dense in $\mathbb{Q}$? In $\mathbb{R}$? Does one imply the other? Recall that A is dense in B if $A \subset B \subset \overline{A}$.

Solution

To show the set of dyadic rationals, $\mathcal{S}$, is dense in $\mathbb{Q}$, it suffices to show that $\mathbb{Q} \subset \overline{\mathcal{S}}$ (dyadic rational are, of course, rationals and so $\mathcal{S} \subset \mathbb{Q}$). Pick any $q \in \mathbb{Q}$. If q is a dyadic rational then the sequence $(q, q, \ldots)$ shows that it is already in $\overline{\mathcal{S}}$. Otherwise consider a sequence $(s_n)_{n \geqslant 0} \in \mathcal{S} \subset \mathbb{Q}$ with $s_0 = \lfloor q \rfloor$ (the floor function) and $s_i = f(i)/2^i$ where $f(i)$ is defined as the following:

$$f(i) = \begin{cases} -1 & \text{if } \sum_{i=0}^{i-1} s_i > q \\ 1 & \text{if } \sum_{i=0}^{i-1} s_i < q \end{cases}.$$

For example, the number $8.76\dot{5}$ would be followed by the sequence $(8, 8.5, 8.75, 8.625, 8.6875, \dots)$. This is a Cauchy sequence as the distance between the s_{i-1} and s_i is $1/2^i$, which clearly converges to 0. Since this is a sequence in $\mathbb{R}$ we know that it converges to q. Therefore $q \in \mathbb{Q} \implies q \in \overline{\mathcal{S}}$, so $\mathbb{Q} \subset \overline{\mathcal{S}}$; hence $\mathcal{S}$ is dense in $\mathbb{Q}$.

From this we conclude that being dense in $\mathbb{Q}$ implies being dense in $\mathbb{R}$, notice that

$$\overline{\mathcal{S}} \supset \mathbb{Q} \implies \overline{\overline{\mathcal{S}}} \supset \overline{\mathbb{Q}} \implies \overline{\mathcal{S}} \supset \mathbb{R}.$$

This tells us $\mathcal{S}$ is not only dense in $\mathbb{Q}$ but also $\mathbb{R}$.

On the other hand, if $\mathcal{S}$ is a subset of $\mathbb{Q}$ and is dense in $\mathbb{R}$, then

$$\overline{\mathcal{S}} \supset \mathbb{R} \supset \mathbb{Q} \implies \overline{\mathcal{S}} \supset \mathbb{Q}$$

which shows that $\mathcal{S} \subset \mathbb{Q}$ is also dense in $\mathbb{Q}$.

Problem 5: 2.98 (Pugh)

Show that $S \subset M$ is somewhere dense in M if and only if $\operatorname{int}(\overline{S}) \neq \varnothing$. Equivalently, S is nowhere dense in M if and only if its closure has empty interior.

Solution

We first show the $\implies$ direction and suppose that $S \subset M$ is somewhere dense. Then there exists some open, nonempty $U \subset M$ such that $S \cap U$ is dense in U. Pick any $x \in U$. Then since $\overline{S \cap U} \supset U$, there has to exist a sequence $(p_n) \in S \cap U$ that converges to x. Clearly this sequence lies in S, which shows $x \in \overline{S}$ and so $U \subset \overline{S}$. The openness of U suggests that the interior of S cannot be empty, for there exists some $B_r(x) \subset S$ which makes x an interior point.

For the converse, suppose $\operatorname{int}(\overline{S})$ is nonempty. Then there exists some $p \in \operatorname{int}(\overline{S})$ and some $r > 0$ such that $B_r(p) \subset \overline{S}$. Pick any $q \in B_r(p)$ and we immediately know $q \in \overline{S}$, namely there exists some $(q_n) \in S$ converging to q. Therefore, there exists some $N \in \mathbb{N}$ satisfying if $n \geqslant N$ then

$$d_M(q_n, q) < r - d_M(p, q).$$

By triangle inequality, this implies

$$d_M(q_n, p) \leqslant d_M(q_n, q) + d_M(p, q) = r$$

and so if $n \geqslant N$ we know $q_n \in B_r(p)$. Therefore q also happens to be a limit point of $B_r(p)$. Thus $q \in \overline{S \cap B_r(p)}$. Since q is arbitrary, we conclude that $B_r(p) \subset \overline{S \cap B_r(p)}$, and thus S is dense in $B_r(p)$[1].

[1] Linfeng sketched this proof during his office hour on Oct. 18.

Problem 6: 2.99 (Pugh)

Let M, N be nonempty metric spaces and $P = M \times N$.

(1) If M, N are perfect, prove that P is perfect.

(2) If M, N are totally disconnected, prove that P is totally disconnected.

(3) What about the converses?

Solution

(1) Pick any $p = (m, n) \in P$. Since M, N are perfect we know there exist sequences $(m_k) \in M$ and $(n_k) \in N$ that converge to $m \in M$ and $n \in N$, respectively. Therefore the sequence $(m_k, n_k) \in P$ converges to (m, n) and p is a cluster point. Hence P is perfect.

(2) Pick any connected $S \subset P$. We want to show that S must be a singleton. Consider the projection functions $P \to M$ and $P \to N$ defined by $(m, n) \mapsto m$ and $(m, n) \mapsto n$. Since these functions are continuous, we know that both images of S must be connected. By the total disconnectedness of M and N they have to be singletons $\{m_0\} \subset M$ and $\{n_0\} \subset N$. Therefore $S = \{(m_0, n_0)\}$, also a singleton in P. This concludes the proof that P is indeed totally disconnected.

(3) The converge is not true in general[2]. If we let M be a singleton $\{m\}$ and N be $\mathbb{R}$, then the product $\{m\} \times \mathbb{R}$ is perfect [any $(p_n) \to p$ also satisfies $((p_n), m) \to (p, m)$] while $\mathbb{R}$ is not.

For total disconnectedness, suppose P is totally disconnected. We start by inspecting M. Let $S \subset M$ be connected. Consider the function $f : m \mapsto (m, n_0)$ for some $n_0 \in N$. Since the continuous image of a connected set is connected, we know $f(S)$ is connected. On the other hand, the total disconnectedness of P implies $f(S)$ has to be a singleton, and therefore $S = f^{-1}(f(S))$ is also a singleton. This shows M is totally disconnected. The proof showing N is totally disconnected is analogous and is omitted. To sum up, P being totally disconnected implies M and N both being totally disconnected.

Problem 7

(1) Prove that a real number lies in the Cantor set $\mathcal{C}$ if and only if it has a ternary (i.e., base 3) expansion without any 1's.

(2) Is every number in the Cantor set an endpoint of one of the intervals we removed? Is every number in

[2] Thanks to Bruno again for pointing out that if (m, n) converges, it's not necessarily true that there exist $(m_k) \to m$ and $(n_k) \to n$, both of which not being the constant sequence.

the Cantor set rational?

Solution

(1) The $\implies$ direction is obvious. Pick any $x \in \mathcal{C}$ that's expressed by its ternary expansion. Clearly $x \in C^1$. Since C^1 excludes $(1/3, 2/3)$, this rules out the possibility of x's first (decimal) digit's being 1. Then since C^2 excludes the middle thirds of each intervals of C^1, we know that the second digit of x cannot be 1, either. Since the way to construct each C^n is analogous, we conclude that the ternary expansion of x does not have any 1's.

For $\impliedby$, we want to show that everything of form

$$\sum_{i=1}^{\infty} \frac{a_i}{3^i} \text{ with } a_i \in \{0, 2\},$$

i.e., the ternary expansion without any 1's, is indeed in $\mathcal{C}$. The key thing to to notice here is that the sequence

$$(x_k) = (x_1, x_2, x_3, \dots) := (\sum_{i=1}^{1} \frac{a_i}{3^i}, \sum_{i=1}^{2} \frac{a_i}{3^i}, \sum_{i=1}^{3} \frac{a_i}{3^i}, \dots)$$

is Cauchy in $\mathbb{R}$ and thus convergent. Denote intervals of C^n by I_n. Induction — if not inspection — suggests that each partial sum, i.e., each x_n, is the left endpoint of some $I'_n \in C^n$. [We don't know precisely which interval it will be because that depends on whether the digits are 0 or 2, but we know it belongs to *some* interval in C^n.] Then, since

$$\sum_{i=n+1}^{\infty} \frac{a_i}{3^i} \leqslant \sum_{i=n+1}^{\infty} \frac{2}{3^i} = \frac{1}{3^n} \text{ and } I'_n \text{ has length } \frac{1}{3^n}$$

we know that $m \geqslant n \implies a_m \in I'_n$. Therefore, since I'_n is closed, it contains all its limits, and so $\lim(x_m)_{m \geqslant n} = \sum_{i=1}^{\infty}(a_i)/(3^i) \in I'_n$. Since n is chosen arbitrarily, this limit actually lies in all I'_k for all $k \in \mathbb{N}$, i.e.,

$$\lim_{k \to \infty}(x_k) = \sum_{i=1}^{\infty} \frac{a_i}{3^i} = \bigcap_{i=1}^{\infty} I'_i \in \bigcap_{i=1}^{\infty} C^i = \mathcal{C}.$$

(2) No and no. For the first no, consider

$$\sum_{i=1}^{\infty} \frac{2}{3^{2i}} = \frac{2}{9} \cdot \frac{9}{8} = \frac{1}{4} = (0.\dot{0}\dot{2})_{\text{base } 3}$$

which is in $\mathcal{C}$ but not an endpoint since all endpoint are 3-adic numbers, i.e., with denominators being powers of 3. Suppose this were an endpoint of form $k/3^n$, then $4k = 3^n$, clearly a contradiction.

Alternatively, we can simply use the fact from the following part to say that, there are irrational numbers in $\mathcal{C}$ whereas each endpoint represents a rational number.

The second no is even more blatant — we know $\mathcal{C}$ is uncountable but $[0,1]$ only contains countably many rationals! For an example, consider the following ternary decimal which is nowhere periodic — thus not rational — yet following a easy-to-spot pattern:

$$(0.20\,200\,2000\,20000\ldots)_3 = \sum_{i=1}^{\infty} \frac{2}{3^{i(i+1)/2}}.$$

Problem 8

Write out a reasonable and precise definition of the *middle fifths Cantor set*. Convince yourself that this is a Cantor space but you do not need to write it down.

Solution

Let $C^0 = [0,1]$. Removing the middle fifth of C^0 gives two disjoint closed intervals. Denote their union as C^1, i.e., $C^1 := [0,2/5] \cup [3/5,1]$. Now repeat the same process for each interval in C^1, and denote the union of the four smaller intervals of lengths $4/25$ as C^2. So on and so forth. Then the *middle fifths Cantor set* is the nested intersection

$$\mathcal{C}_{\text{fifths}} := \bigcap_{n=0}^{\infty} C^n.$$

Problem 9: (extra credit) 2.68 (Pugh)

List the closed convex sets in $\mathbb{R}^2$ up to homeomorphism. There are nine. How many are compact?

Solution

The nine "types" of sets are: $\varnothing$ the empty set; (x,y) a point; $\{x\} \times [a,b]$ a line segment; $\{x\} \times [0,\infty)$ a ray[3], $\{x\} \times \mathbb{R}$ a line; D_1 a closed disk; $[a,b] \times \mathbb{R}$ a strip; $[0,\infty) \times \mathbb{R}$ a closed half plane; and $\mathbb{R}^2$ the entire plane.

By Heine-Borel theorem, the compact ones are closed and bounded. This corresponds to the empty set, the point, the line segment, and the closed disk.

[3] I talked with Prof. Andrew Manion [the one teaching 425b this spring]about this one. He reminded me that rays and lines aren't homeomorphic yet they are bost closed convex sets in $\mathbb{R}^2$.

Problem 10: *fake extra credit* 2.31 (Pugh)

Prove the following:

(1) If U is an open subset of $\mathbb{R}$ then it consists of countably many disjoint [open?] intervals $U = \coprod U_i$. Unbounded intervals $(-\infty, b), (a, \infty)$, and $(-\infty, \infty)$ are permitted.

(2) These intervals U_i are uniquely determined by U. In other words, there is only one way to express U as a disjoint union of open intervals.

(3) If $U, V \in \mathbb{R}$ are both open so that $U = \coprod U_i$ and $V = \coprod V_j$ where U_i and V_j are open intervals, then $U \cong V$ if and only if there are equally many U_i's and V_j's.

Solution

(1) Let U be an open set and pick $x \in U$. Consider the following sets:

$$A_x := \{a \in \mathbb{R} \mid a < x \text{ and } t \in U \text{ for all } t \in (a, x)\}$$

and

$$B_x := \{b \in \mathbb{R} \mid b > x \text{ and } t \in U \text{ for all } t \in (x, b)\}.$$

Since U is open and $x \in U$ we know that there exists some $\epsilon > 0$ satisfying $(x - \epsilon, x + \epsilon) \subset U$. Therefore A and B are nonempty. Define $\inf(A) = -\infty$ if A is not bounded from below and $\sup(B) = \infty$ if it is not bounded from above. Other than these extreme cases we get $\inf(A) = m$ and $\sup(B) = n$ by the LUB property. Then we claim (m, n) is the "largest"[4] interval containing x.

Notice that

$$U' := \bigcup_{x \in U} \big(\inf(A_x), \sup(B_x)\big) = U.$$

$U' \subset U$ is immediate since each $\big(\inf(A_x), \sup(B_x)\big)$ is a subset of U. On the other hand, if $x \in U$ then $x \in \big(\inf(A_x), \sup(B_x)\big) \subset U'$. Hence $U' = U$.

Now we will show that these intervals are disjoint: suppose $\inf(A_x) < \inf(A_y) < \sup(B_x) < \sup(B_y)$ and $\big(\inf(A_x), \sup(B_y)\big)$ forms an interval. Then it immediately follows that $\sup(B_y)$ would be the supremum of B_x and $\inf(A_x)$ would be the infimum of A_y, contradiction. Hence for $x \neq y$, either $\big(\inf(A_x), \sup(B_x)\big) = \big(\inf(A_y), \sup(B_y)\big)$ or $\big(\inf(A_x), \sup(B_x)\big) \cap \big(\inf(A_y), \sup(B_y)\big) = \varnothing$. Since sets ignore duplicate values, the intervals of U' — and also U — are disjoint.

Now it remains to show that U' consists of countably many intervals. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, each interval contains *at least* one rational number. Since rationals are countable, the number of intervals must also be countable, and this concludes our proof.

(2) Suppose that

$$U = \coprod_{i=1}^{m} U_i = \coprod_{j=1}^{n} V_j.$$

where U_i's and V_j's are open. [Not to be confused with part (3), here we are simply assuming that the disjoint unions of U_i's and of V_j's are both equal to U.] Now fix U_1 and consider the following equality:

$$U_1 = \bigcup_{j=1}^{n} (U_1 \cap V_j).$$

Clearly U_1 is connected, while on the other hand each $U_1 \cap V_j$ is disjoint from each other since the V_j's are disjoint. This means all but one intersection are empty, and the nonempty one satisfies $U_1 \cap V_{j_1}$ for some $1 \leqslant j_1 \leqslant n$. Therefore we have $U_1 \subset V_{j_1}$.

On the other hand, we can apply the same logic and argue that V_{j_1} is a subset of some U_1'. Then we have $U_1 \subset V_{j_1} \subset U_1'$, the only possibility being $U_1 = V_{j_1} = U_1'$ since the U_i's are disjoint and it's impossible that U_1 is a subset of some disjoint interval.

Likewise, we can show that $U_2 = V_{j_2}$ and so on, and eventually we would get to the conclusion that $i = j$ and $\{U_i\} = \{V_j\}$, i.e., the representation of U is unique.

(3) This immediate from the fact that the preimage of (a', b') under a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is an open interval (a, b).

For the $\Longleftarrow$ direction, we can set up a one-to-one correspondence between the intervals of U_i's and V_j's and define a homeomprhism between each pair of intervals:

$$f : (a, b) \to (a', b') \text{ defined by } x \mapsto a' + \frac{x - a}{b - a}(b' - a').$$

For $\Longrightarrow$, suppose, by contradiction, that $|\{U_i\}| \neq |\{V_j\}|$. WLOG assume $u := |\{U_i\}| < |\{V_j\}| := v$. We know that the preimage of $V_1, \ldots, V_u$ must all be open intervals, and that exhausts $\{U_i\}$. Yet we still have $v - u$ open intervals in $\{V_j\}$ that don't get matched to a open preimage (clearly $\varnothing$ does not count). This means a homeomorphism cannot exist, contradiction. Hence $|\{U_i\}| = |\{V_j\}|$.

Problem 10: (extra credit) 3.31 (Pugh)

Consider the "middle fourth Cantor set" by each time removing the middle interval of length $1/4^n$ in the n^{th} iteration. Denote each set of intervals after n iterations as F^n.

(1) Prove that $F = \bigcup F^n$ is a Cantor set but not a null set. It is referred to as a **fat Cantor set**.

(2) Infer that being a zero set is not a topological property. If two sets are homeomorphic and one is a null set then the other need not be a zero set.

Solution

For the first part, I'm assuming that we are using the following definition:

> We say M is a **Cantor Space** if, like the standard Cantor set $\mathcal{C}$, it is compact, nonempty, perfect, and totally disconnected.

(1) The compactness of F comes from the fact that F is the union of intersections of compact sets (closed intervals) and is therefore compact. It is nonempty — observe that $0, 1 \in F$.

Before we move to perfectness and totally disconnectedness, notice that the main difference between $\mathcal{C}$ and F is that the lengths of intervals in F^n are no longer $1/3^n$ but is something more complicated. "Inspection" suggests that, after the first iteration, intervals have length $3/8$; after the second, intervals have length $5/32$, and after the third, $9/128$, and so on. More generally, the length of intervals after n^{th}

[4]Credit to Prof. Manion again for reminding me that the existence of "largest" intervals should be justified using supremum and infimum.

iteration can be described as

$$L(n) = \frac{2^n + 1}{2^{2n+1}}.$$

A simple calculation suggests that

$$\lim_{n\to\infty} L(n) = \lim_{n\to\infty} \frac{2^n + 1}{2^{2n+1}} = \lim_{n\to\infty} \frac{2^n}{2^{2n+1}} + \lim_{n\to\infty} \frac{1}{2^{2n+1}} = 0 + 0 = 0.$$

Having shown this, we can proceed with the proof of F's perfectness and total disconnectedness.

Pick any $f \in F$ and any $\epsilon > 0$. Then there exists $n \in \mathbb{N}$ large enough such that intervals in F^n is smaller than ϵ. This tells us that the point f lies in one of the 2^n intervals in F^n, which we denote as I, and we also know I is completely contained in the neighborhood $(f - \epsilon, f + \epsilon)$. Therefore F clusters at f and so F is perfect.

For total disconnectedness, again recall the proof related to $\mathcal{C}$. Since I is closed in $\mathbb{R}$, it is also closed in F^n. The complement $F^n \setminus I$ is the union of finitely many closed intervals and is also closed. Hence I and $F^n \setminus I$ are clopen in F^n. By the inheritance principle, $F \cap I$ is a clopen neighborhood of $f \in F$ contained entirely in $(f - \epsilon, f + \epsilon)$. This shows F is totally disconnected.

Now, to show that F has positive outer measure, notice that the total length of intervals removed is the infinite sum

$$\frac{1}{4} + 2 \cdot \frac{1}{16} + \cdots = \sum_{i=1}^{\infty} 2^{i-1} \frac{1}{4^i} = \frac{1}{4} \cdot 2 = \frac{1}{2}.$$

This means the total remaining length is $1 - 1/2 = 1/2 > 0$ which shows that F is not a null set.

(2) This follows from Moore-Kline theorem.