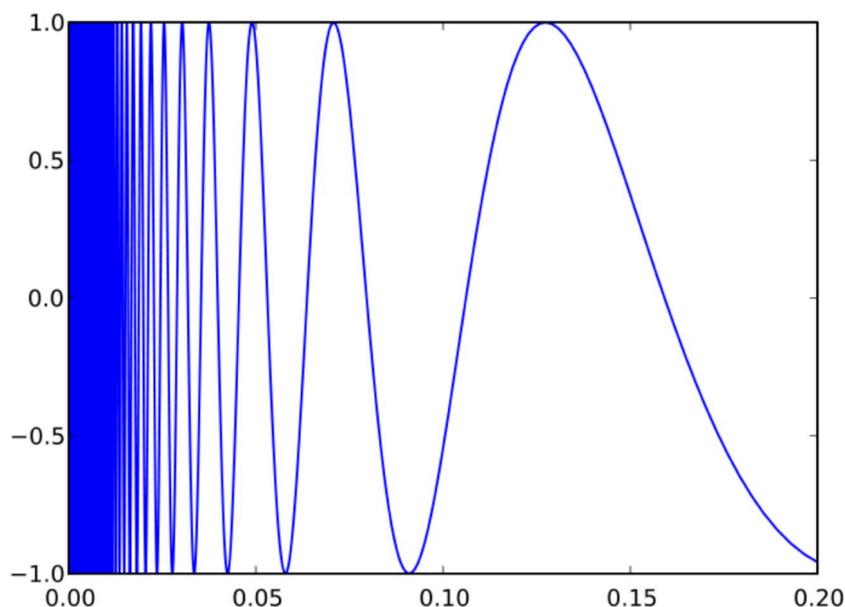


6 Wed 9/23

Recall from last class: definition of open and closed subsets: open if the subset contains all limits, and closed if the subset satisfies that $M_r(p)/B_r(p)$ condition — that the open subset has no boundary and we can always fit a tiny “ball” around a point in the subset no matter how close it is to the “boundary”.

Example 6.1: Topologist’s Sine Curve

Consider $M = \mathbb{R}^2$ and $S = \{(x, \sin(1/x)) \mid x \in (0, 1]\} \cup \{(0, 0)\}$. The graph is a sinusoidal wave but with ever increasing frequency as $x \rightarrow 0$.



Topologist’s Sine Curve, from [class website](#)

The set is not open because $(x, \sin(1/x) + \epsilon) \notin S$ no matter how small ϵ is. The graph of this set is a line and it has lower dimension than \mathbb{R}^2 .

Also consider all the points of interceptions between the graph of S and $y = 1$. The points converge to $(0, 1)$ but clearly $(0, 1) \notin S$.

To make this set closed, consider revising it as

$$S = \{(x, \sin(\frac{1}{x})) \mid x \in (0, 1]\} \cup \{(0, y) \mid y \in [-1, 1]\}.$$

Definition 17

Given any subset $S \subset M$, its **closure**, denoted by \overline{S} , is given by the set of all limits of S .

Remark

This set is closed. In particular, this is the *smallest* closed subset of M containing S .

Example 6.2

$$\overline{\mathbb{Q}} = \mathbb{R}.$$

Now we will give a proof for a theorem mentioned in last class:

Theorem 18

Openness is dual to closedness.

Proof

Suppose $S \subset M$ is an open subset. We want to show that $S^c := M \setminus S$ is closed. Let (p_n) be a sequence in S^c . It has to converge to some $p \in M$. We want to show that $p \in S^c$. Suppose not, then $p \in S$, and by the openness of S and the convergence of (p_n) we have

$$\text{there exists } \epsilon > 0 \text{ and } m \in \mathbb{N} \text{ such that } \begin{cases} d(p_m, p) < \epsilon \text{ (convergence of } (p_n)) \\ p_m \in M_\epsilon(p) \implies p_m \in S \text{ (openness)} \end{cases}$$

and this contradicts our assumption that $(p_n) \in S^c$. Hence S is closed.

Now suppose $S \subset M$ is closed. We want to show S^c is open. By definition we want to show that for all $p \in S^c$ there exists $\epsilon > 0$ such that $M_\epsilon p \subset S^c$. Suppose not, then for some point $p \in S^c$, for all $\epsilon > 0$, we always have $M_\epsilon p \cap S \neq \emptyset$ since we cannot find any balls enclosing p that's completely in S^c . Then if we set $\epsilon = 1/n$ with $n = 1, 2, \dots$, we can find a sequence of points from $M_\epsilon p \cap S$ as ϵ changes. Then this sequence converges to p since $(1, 1/2, 1/3, \dots)$ converges to 0. However, $p \in S^c$, so S does not contain all limit points. Contradiction. Therefore S^c must be open. \square