1 Tue 9/29 Discussion

Definition 1

A topology on M is the collection $\mathcal{T}$ of subsets of M satisfying

(1) $\varnothing, M \in \mathcal{T}$,

(2) arbitrary union of sets in $\mathcal{T}$ is in $\mathcal{T}$, and

(3) finite intersection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

Example 1.1

Let $M = \{1,2,3\}$ and define

$$\mathcal{T}(M) = \{\varnothing, \{1\}, \{2,3\}, \{1,2,3\}\}.$$

Theorem 2

Given (M, d), a metric space, the collection of open sets forms a topology on M. (This is immediate from what we've done last class.)

Remark

Given a metric space, we get a topology *induced by the metric*. Such topology is called **metrizable**.

Example 1.2

The example above with $M = \{,1,2,3\}$ and

$$\mathcal{T}(M) = \{\varnothing, \{1\}, \{2,3\}, \{1,2,3\}\}$$

is not a metrizable topology. It is not induced by any metric because any singleton in a metric is closed, so $\{3\}$ is closed and $\{1,2\}$ is open which is not in $\mathcal{T}(M)$.

Corollary 3

Given (M, d), the following are equivalent:

(1) arbitrary union of open sets is open in (M, d),

(2) finite intersection of open sets is open in (M, d),

(3) arbitrary intersection of closed sets is closed in (M, d), and

(4) finite union of closed sets is closed in (M, d).

Definition 4

Let $(M, \mathcal{T}_M)$ and $(N, \mathcal{T}_N)$ be topological spaces (set with topology), we say $f : M \to N$ is **topologically continuous** if the $E \in \mathcal{T}_N$ is open then $f^{-1}(E) \in \mathcal{T}_M$ is also open.

Theorem 5

The following are equivalent for continuity of $f : M \to N$:

(1) f preserves sequential continuity.

(2) f satisfies the $\epsilon - \delta$ condition.

(3) f is topologically continuous:

  (1) the preimage of an open set is open, or

  (2) the preimage of a closed set is closed.

A quick recap of the inheritance principle:

Inheritence Principle

Let (M, d) be a metric space, and $N \subset M$ a subspace with the induced metric. A subset $U \subset N$ is open (in N) if and only if $U = N \cap V$ for some open subset $V \subset M$. Likewise for the condition to be closed.

Example 1.3

Let $M = \mathbb{R}$ and d the standard Euclidean metric. Let $N = [0, 1)$. Is it open in M? In N?

Solution

Not in M because no ball containing 0 is also a subset of this interval. However, it is open in N because

$$[0, \frac{1}{2}) = \underbrace{(-1, \frac{1}{2})}_{\text{open subset of } \mathbb{Q}} \cap \underbrace{[0, 1)}_{N \text{ itself, open}}$$

Quick recap of connectedness:

Definition 7

Let (M, d) be a metric space. M is called **disconnected** if M can be written as $M = A \sqcup A^c$ where A and A^c are proper clopen sets.

Example 1.4

$M = (-1, 1) \cup (2, 3)$ with standard metric is disconnected.

If we let $A = (-1, 1)$, then A is open in M and A^c is closed. Likewise, A^c is open and A is closed. Hence A, A^c are proper clopen sets in M.

Example 1.5

$\mathbb{Q}$ is disconnected because it can be written as

$$\mathbb{Q} = \{r \in \mathbb{Q} \mid r < \sqrt{2}\} \cup \{r \in \mathbb{Q} \mid r > \sqrt{2}\}.$$

Definition 8

(M, d) is called **connected** if it is not disconnected.

Theorem 9

If $f : M \to N$ is continuous and surjective, then if M is connected we have N is connected See Pugh p.87.

Worksheet # 7

Problem 1

Give a metric space $(M.d)$ and $S \subset M$, define

$$\lim(S) = \{p \in M \mid p \text{ is a limit point of } S\}.$$

Show that

$$\lim(S) = \bigcap_{V \supset S, V \text{ closed}} V.$$

Note that the RHS is called the closure of S in topology.

Solution

We will show mutual inclusion. First, LHS $\subset$ RHS. Pick $p \in \lim(S)$. By definition there exists a sequence $(p_n) \to p$. Since $S \subset V$, we know (p_n) is also in V. Since each V is closed, we know $p \in \bigcup V$.

Now to show RHS $\subset$ LHS: first notice that $\lim(S)$ is closed (see Pugh p.68), and we know $S \subset \lim(S)$ since for any element $s \in S$, the sequence $(s, s, \dots)$ converges to $s \in \lim(S)$. Now notice that $\lim(S)$ meets the requirement to be a V, and let's call it V_i. Hence $\bigcap V \subset V_i = \lim(S)$.

Problem 2

Prove or disprove the following:

(1) If A and B are connected, then so is $A \cap B$.

(2) If A and B are connected, then so is $A \cup B$.

Solution

(1) No. Consider $\{(x,y) \mid y=0\} \cup \{(x,y) \mid x^2+y^2=1\}$.

(2) Even more obvious. No.

Remark

A metric space (M,d) with finitely many elements is disconnected.

Problem 3

Definition: given metric space (M,d), a subset $E \subset M$ is called **dense** if for all $x \in M$, either $x \in E$ or x is a limit point of E.

Let f be a continuous function from one metric space (M,d_M) to another metric space (N,d_N), and let E be a dense subset of M. Prove that $f(E)$ is dense in $f(M)$.

Remark

To visualize "dense", think of $\mathbb{Q}$ in $\mathbb{R}$. Pick $x \in \mathbb{R}$. It is either rational or irrational, namely $x \in \mathbb{Q}$ or $x \in \mathbb{R} \setminus \mathbb{Q}$. If it is irrational, we can always find a rational within $(x, x+1/n)$. Then x becomes a limit point.

Solution

We want tos how that for any $y \in f(M)$, either $y \in f(E)$ or y is a limit poinf of $f(E)$.

Now for such $y \in f(M)$, there exists $x \in M$ such that $f(x)=y$. Since E is dense in M, either $x \in E$ or x is a limit point of E.

(1) If $x \in E$ then $y \in f(E)$ and we are done.

(2) If x is a limit point of E then there exists $(x_n) \to x$. By the sequential continuity of f we know $f(x_n) \to f(x) = y$. Since each $f(x_i) \in f(E)$ we know y is the limit of a sequence in $f(E)$ and therefore is a limit point of $f(E)$.

2 Wed 9/30

Definition 10: Compactness

A metric space (M, d) is **sequentially compact** if every subsequence in M has a convergent subsequence.

Remark

Later, we will give an equivalent definition for compactness: "covering compact". For now we will focus on sequential compactness.

Definition 11

A metric space (M, d) is **bounded** if there exists $R > 0$ such that $d_M(x, y) < R$ for all $x, y \in M$. In other words, the distance between any two points is bounded.

Lemma 2.1

A compact metric space is bounded.

Proof of lemma

Assume by contradiction that (M, d) is unbounded. Then we can find $y \in M$ and a sequence (x_n) such that $d_M(y, x_i) \geqslant i$. Then the sequence $(y, x_1, x_2, \dots)$ does not have any convergent subsequence. □

Example 2.1

$\mathbb{R}^n$ is not compact.

Example 2.2

If (M, d) is a finite metric space then it is compact because for each sequence we can find a constant sequence.

Definition 12

If (M, d) is a metric space, we say that a subset $S \subset M$ is compact if S with the induced metric is compact.

Lemma 2.2

Any compact subset of a metric space is closed.

Example 2.3

The open interval $(0, 1)$ is not compact. Any sequence converging to 1 fails to converge in this interval, and so are any subsequences.

Proof of Lemma

Assume $S \subset M$ is compact and let (p_n) be a sequence such that $(p_n) \to p \in M$. Since S is compact, we know that there exists a subsequence (p_{n_i}) converges to $s \in S$. But we also know the subsequence must converge to the same limit as does the mother sequence, we know $p = s$ and $p \in M$. Hence S is closed. □

Problem 4

Does closed and bounded necessarily imply compact?

Heine-Borel Theorem

Any closed and bounded subset of $\mathbb{R}^m$ is compact.

Solution

In general, closed and boundedness do not imply compactness. Consider $\mathbb{N}$ equipped with the discrete metric. Since the distance in a discrete metric is at most 1, every subset is bounded. Let $S \subset \mathbb{N}$ be the closed subset. The sequence $(1, 2, \dots)$ cannot possibly have a convergent subsequence so $\mathbb{R}$ equipped with discrete metric cannot be compact.

Theorem 14

$[a, b]$ (with standard metric) is compact.

Proof

Pick an (infinite) sequence (x_n). First divide $[a, b]$ into two equal halves with $L_1 = [a, a + (b - a)/2]$ and $R_1 = [a+(b-a)/2, b]$. Since (x_n) is infinite, at least one of these intervals contain infinitely many terms of this sequence. Choose an interval that contains infinitely many terms, call it I_1, and proceed with this bisection to get $I_2, I_3, \ldots$. Now we can construct a (sub)sequence (x_{n_k}) such that $x_{n_i} \in I_i$. Then for $i, j \geqslant N$, we have

$$|x_{n_i} - x_{n_j}| \leqslant \frac{b-a}{2^N}$$

which shows that the subsequence (x_{n_k}) is Cauchy. Since $\mathbb{R}$ is complete, it converges in $\mathbb{R}$. Therefore $[a, b]$ is compact. □

Remark

Pugh gives an alternate proof in his book. Pg. 79.

Theorem 15

The Cartesian product of two compact sets is compact. (This can be generalized by induction.)

Proof

Let $A \subset M$ and $B \subset N$ be given. Pick $(a_n) \in A$ and $(b_n) \in B$. The compactness of A implies (a_n) has a convergent subsequence (a_{n_k}) that converges to $a \in A$. Fix these indexes. It's obvious that (b_{n_k}) is also a sequence in B, so it must also have a subsequence $(b_{n_k(\ell)})$ that converges to $b \in B$. Now $(a_{n_k(\ell)})$ is a subsequence of the convergent sequence (a_{n_k}) and must also converge to a. Then

$$(a_{n_k(\ell)}, b_{n_k(\ell)}) \to (a, b)$$

and this shows $A \times B$ is compact. □

3 Fri 10/2

Definition 16

A subset $S \subset M$ of a metric space (M, d) is compact if every sequence in S has a subsequence which converges to a limit in S.

Recall from last class that we've shown that

(1) compact $\implies$ closed and bounded, but

(2) closed and bounded $\not\Longrightarrow$ in general.

(3) However, Heine-Borel Thm states that closed and bounded in $\mathbb{R}^n \implies$ compact.

Alterate proof to show closed intervals are compact

Suppose we are trying to show $[a, b]$ is compact. Let (p_n) be a sequence in $[a, b]$. Define

$$C := \{x \in [a, b] \mid p_i < x \text{ for only finitely many } i\text{'s}\}$$

Observe that C is nonempty ($a \in C$) and it is clearly bounded above by b. Now we want to show (p_n) has a subsequence that converges to $m = \sup(C)$. If not, then there exists $\epsilon > 0$ such that $p_k \in (m - \epsilon, m + \epsilon)$ only finitely many times (suppose not then we can construct a sequence of ϵ's and eventually find a subsequence converging to m). Therefore $m + \epsilon$ is also in C, contradicting m being the supremum. Hence each $(p_n) \in [a, b]$ has a convergent subsequence, and this interval is compact. □

Definition 17: Product metrics

Let (M, d_M) and (N, d_N) be metric spaces. We have the following metrics:

(1) $d_{\text{sum}}((m,n),(m',n')) = d_M(m,n') + d_N(n,n')$. Example: $\mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}$ with

$$d((x,y),(x',y')) = |x-x'| + |y-y'|.$$

(2) $d_E((m,n),(m',n'))\sqrt{(d_M(m,m')^2 + d_N(n,n')^2}$. Example: Euclidean distance.

(3) $d_{\max}((m,n),(m',n')) = \max\{d_M(m,m'), d_N(n,n'\}$.

Theorem 18

Let $((m_k, n_k))$ be a sequence in $M \times N$. The ollowing are equivalent:

(1) the sequence converges with respect to d_{sum};

(2) ... with respect to d_E;

(3) ... with respect to $d_{\max}$;

(4) (m_k) converges in M and (n_k) converges in N.

Theorem 19

All three of these metrics are "topologically equivalent", i.e., they induce the same collection of open subsets of $M \times N$.

Recall that the L_p norm on $\mathbb{R}^n$ is given by

$$\|\mathbf{r}\|_p = \left(|r_1|^p + |_2|^p + \cdots + |r_n|^p\right)^{1/p}.$$

We also put L_∞ as

$$\|\mathbf{r}\|_\infty = \max\{|r_1|, |r_2|, \ldots, |r_n|\}.$$

Remark

All these different L_p norms induce different metrics, but they are indeed topologically equivalent. One sequence with respect to L_{p_1} converges if and only if the sequence also converges with respect to L_{p_2}.

Corollary 20

The product of any finite number of compact metric spaces is compact.

Proof

See the last proof (of sub-sub-subsequences) from class on Wed. Then use induction. □

Corollary 21

Boxes ($[a_1, b_1] \times [a_2, b_2] \times [a_n, b_n]$) are compact.

Theorem 22

Any closed subset of a compact metric space is compact.

Proof

Consider $S \subset M$ (from now on we will drop the cumbersome (M, d) unless it's necessary to explicitly address it) where M is compact, and S is closed. Consider $(p_n) \in S$. By compactness of M, we know a subsequence converges to $p \in M$. But since S is closed, $p \in S$. Hence S is compact. □

Proof of Heine-Borel Thorem

Since S is closed and bounded in $\mathbb{R}^n$. We know $\mathbb{R}^n$ is compact because it is the Cartesian product of n compact metric spaces. Then S is a closed subset of this compact metric space and is therefore compact. □

4 Mon 10/5

Problem 5

Consider a sequence $(p_n) \in \mathbb{R}^2$ defined by $p_k = (\cos^2(k), 3\cos(k) + \sin^3(7k)$. Does it have a convergent subsequence?

Solution

The sequence is contained in the box $[0,1] \times [-4,4]$. By Heine-Borel, the box is compact, so the sequence must have a convergent subsequence.

Bolzano-Weierstrass Theorem

Any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Theorem 24

The image of a compact set under a continuous function is compact.

Proof

Consider $f : M \to N$, two metric spaces, with (M, d_M) compact. We want to show that $f(M)$ is sequentially compact.

Consider a sequence (p_n) in M; it suffices to show that $(f(p_n))$ has a convergent sequence. This is immediate since (p_n) has a convergent by the compactness of M and that f preserves sequential convergence. Hence the image of that subsequence, a subseqeuence of $(f(p_n))$, converges in N. $\square$

Corollary 25

Let (M, d) be a compact metric space. Then any continuous function $f : M \to \mathbb{R}$ assums its minimum and also its maximum.

Proof

The image is $f(M) \subset \mathbb{R}$. By the theorem above, M is compact, so $f(M)$ is a compact subset of $\mathbb{R}$. Recall that compact $\implies$ closed and bounded. Set $A := \inf(f(M))$ and $B := \sup(f(M))$. Since $f(M)$ is closed, it contains all its limit points, and thus $A, B \in f(M)$. $\square$

Remark

We are only saying that f continuous and M compact together imply $f(M)$ is compact. Not the other way around.

(1) Non-compact to non-compact: the identity function $f : \mathbb{R} \to \mathbb{R}$, or the Signoid function $f(x) = e^x/(1+e^x)$

(2) Non-compact to compact: the sine function $f : \mathbb{R} \to [-1,1]$ by $x \mapsto \sin(x)$.

(3) Compact, but not continuous, to non-compact: the piecewise function $f : [-\pi/2, \pi/2] \to \mathbb{R}$:

$$f(x) = \begin{cases} 0 & x = -\frac{\pi}{2}, \frac{\pi}{2} \\ \tan^{-1}(x) & \text{otherwise} \end{cases}$$

Theorem 26

Suppose $f : M \to N$ is continuous. If M is compact, and if f is a bijection, then it's a homeomorphism.

Example 4.1: Non-example

If M is not compact this can be blatantly false: let S^1 be the unit circle in $\mathbb{R}2$. Consider $f : [0, 2\pi) \to S^1$ defined by $\theta \mapsto (\cos\theta, \sin\theta)$. Clearly this function is continuous and bijective, but the inverse is not continuous, for the inverse of any sequence that approaches $(1,0)$ should converge to 2π, while actually $(1,0)$ corresponds to 0.

Definition 27

A **separation** of M is a decomposition $M = A \sqcup A^c$ where both sets are proper clopen.

Example 4.2

$M = (0,1) \cup (1,2]$. The two sets are a separation of M.

Theorem 28

If $f : M \to N$ is continuous and M is connected, then $f(M)$ is connected.

Proof

Suppose $A \subset N$ is proper clopen, then $f^{-1}(A)$ is proper clopen in M. □

Alternately, suppose $N = A \cup A^c$ is a separation of N. Then Then $M = f^{-1}(A) \cup (f^{-1}(A))^c$ which again shows M is disconnected since this is a separation of M. Hence no separation of N exists. □

5 Tue 10/6 Discussion

Proposition 29

If $f : M \to N$ is a homeomorphism, then for all $p \in M$, $M \smallsetminus \{p\} \cong N \smallsetminus \{f(p)\}$.

Proof

Now we strict f on $M \smallsetminus \{p\}$ and call this g. It's still a bijection because p and its corresponding $g(p)$ are simultaneously removed. Now we want to show that g is open. Pick $E \in N \smallsetminus \{f(p)\}$ that is open. By inheritance principle, since E is open, $E = U \cap (N \smallsetminus \{f(p)\}$ for some U open in N. Then

$$g^{-1}(E) = g^{-1}(U \cap (N \smallsetminus \{f(p)\}) = f^{-1}(U \cap (N \smallsetminus \{f(p)\}) = f^{-1}U \cap f^{-1}(N \smallsetminus \{f(p)\})$$

and by inheritance principle, $g^{-1}(E)$ is again open. Hence g is continuous. Likewise we can show g^{-1} is continuous, which then makes g a homeomorphism. □

Remark

If $M \cong N$, then for finite set $P = \{p_1, p_2, \ldots, p_n\}$, we have

$$M \setminus P \cong N \setminus f(P).$$

Theorem 30

Compactness and connectedness are preserved under homeomorphism, whereas completeness and boundedness are not necessarily preserved under homeomorphism.

Example 5.1

Consider the arctan function betweem $(0,1)$ and $\mathbb{R}$. Clearly $\mathbb{R}$ is not complete but $\mathbb{R}$ is not bounded, yet the arctan function is a homeomorphism.

Definition 31

Let (M, d) be a metric space. $K \subset M$ is called **compact** if for each $(p_n) \in K$, there exists a subsequence converging to a point $p \in K$.

Remark

Being closed and bounded do not necessarily imply being compact. Consider $\mathbb{Z}$ with discrete metric. $\mathbb{N}$ is closed and bounded since the distance between any two elements is at most 1. It is not compact because $(1, 2, \ldots)$ does not have a convergent subsequence. However, closed and bounded imply compact in $\mathbb{R}^n$ by Heine-Borel theorem.

Worksheet 8

Problem 6

Show that any two of $(0,1), (0,1]$, and $[0,1]$ are not homeomorphic.

Solution

To show $(0,1) \ncong [0,1]$, if we remove $\{1\} \in [0,1]$ then we have to remove some $\{x\}$ for some $x \in (0,1)$. Then the first set becomes disconnected while the second is still connected.

Likewise for $(0,1) \ncong (0,1]$ if we take away $\{1\}$ from the second interval.

For $(0,1] \ncong [0,1]$, taking off two points will always finish the proof.

Problem 7

Show that arbitrary intersection of compact sets is compact, and the union of finitely many compact sets is compact.

Solution

Suppose $(K_1, K_2, \dots)$ are compact sets. The compactness implies they are all bounded and closed. Therefore their intersection is also closed. Clearly this intersection is also a subset of compact set. Therefore it's compact.

Pick any sequence in (p_n) in the union. There exists at least one K_i from which comes infinitely many terms that constitute (p_n). Then it's a one line proof by the compactness of K_i.

Problem 8

Determine whether the following sets are compact or not.

(1) $X = \{(x,y) \in \mathbb{R}^2 \mid x \geqslant 0, y = \sin x\}$. *This is unbounded and therefore not compact by Heine-Borel.*

(2) $Y = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, the unit sphere in $\mathbb{R}^3$. *This is compact because it is closed and bounded (by Heine-Borel).*

(3) $Z = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\} \subset \mathbb{R}$. *This is compact by Heine-Borel again.*

Problem 9

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and consider $A \subset R$ and its graph $G = \{(x, f(x)) \mid x \in A\}$. Show that

(1) If A is closed, then G is closed.

(2) If A is compact, then G is compact.

Remark

If A is closed, $f(A)$ is not necessarily closed. Consider the Signoid function $f(x) = e^x/(1+e^x)$ and let $A = [0,\infty)$, a closed interval. Then $f(A) = [\frac{1}{2}, 1)$, not closed.

Solution

(1) We want to show that any convergent sequence in G converges in G. Consider $\{(p_n), (f(p_n))\}$. If it converges, this implies (p_n) converges to p then the continuity of f implies $(f(p_n)) \to fp$. Hence the sequence in G converges to $(p, fp) \in G$. Then G is closed.

(2) A being compact implies A being closed and bounded. On the other hand we know, from the previous part, that G is closed. Now it suffices to show G is bounded. Yet $f(A)$ implies $f(A)$ is bounded. Hence both A bounded and $f(A)$ bounded implies G bounded.

6 Wed 10/7

Example 6.1

Let $A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leqslant x^2+y^2 \leqslant 2\}$. Does there exist a continuous function $f : A \to \mathbb{R}$ which is unbounded from above?

Solution

No. First note that A is closed and bounded in $\mathbb{R}^2$ so by Heine-Borel it is compact. Recall that a continuous function maps compact sets to compact sets, so $f(A)$ must be compact which, in turn , means it is closed and bounded.

Remark

If we define $A = \{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$ then there does exist a continuous function that blows up to infinity.

Theorem 32

If (M, d) is a metric space and $S \subset M$ is a connected subset, then anything betweem S and $\overline{S}$, i.e., any T satisfying $S \subset T \subset \overline{S}$, is connected.

Proof

We prove by contrapositive: that if T is disconnected then S is disconnected. By definition, if T is disconnected then $T = A \sqcup B$ where both A and B are proper clopen. If we define $A' = A \cap S$ and $B' = B \cap S$ then $S = A' \sqcup B'$.

Claim: A', B' are proper, nonempty, clopen subsets of S. By inheritance A' and B' are both clopen. No we'll show that A' and B' are proper and nonempty. Suppose, by contradiction, that A' is empty. Then $A = T \setminus S$. On the other hand, A is by definition open and nonempty. Hence if we pick $p \in A$, there exists $\epsilon > 0$ such that for all $x \in T, d(x, p) < \epsilon \implies x \in A$. But since $p \in \overline{S} \setminus S$, we can find a $y \in S$ satisfying $d(y, p) < \epsilon$. Then $y \in S \cap A$, contradicting the assumption that $A \subset T \setminus S$. Hence A' and B' must be nonempty, proper, and clopen. Therefore S is disconnected. Hence proven the contrapositive. □

Generalized IVT

If (M, d) is a connected metric space and $f : M \to R$ a continuous function, then if $a, b \in f(M)$ and $a < b$, then for all $c \in (a, b)$ we have $c \in f(M)$.

Proof

Suppose, by contradiction, that there exists some $c \in (a,b)$ with $c \notin f(M)$. Then we can come up with a separation of M. Define $A = f^{-1}((-\infty, c))$ and $B = f^{-1}((c, \infty))$. Clearly $M = A \sqcup B$. The two sets are both nonempty. They are both open because f is continuous and thus maps open sets to open preimages. Since they are complements of each other they are also closed. Hence $A \sqcup B$ is a separation of M, contradiction M being connected. □

Theorem 34

$\mathbb{R}$ is connected.

Proof

Suppose $\mathbb{R} = A \sqcup A^c$ where A is proper clopen. By assumption A is not empty; consider $p \in A$. Consider $S = \{x \in \mathbb{R} \mid (p,x) \subset A\}$.

Claim: $(p, \infty) \subset A$. If this claim fails, then S is bounded from above and also nonempty (since A is open and we have the neighborhood argument which shows something $> p$ is also in A). If we define $c = \sup S$ we see that anything less c is contained in S and thus in A. Hence c is a limit of A. Since A is closed, $c \in A$. Again since A is open, there exists $\epsilon > 0$ with $(c - \epsilon, c + \epsilon) \in A$, contradicting $c = \sup S$. Hence $(c, \infty) \subset A$. Likewise $(-\infty, c) \subset A$. Therefore $A = \mathbb{R}$, contradicting its being clopen. Hence $\mathbb{R}$ is connected. □

Proposition 35

$(0, 1)$ is connected since it is homeomorphic to $\mathbb{R}$ (consider arctan with coefficients or the Signoid function). Since connectedness is a topological property, this interval is connected just like $\mathbb{R}$. More generally, (a, b) is also connected.

Proposition 36

$[0, 1]$ is connected: consider $f : \mathbb{R} \to [0, 1]$ defined by $x \mapsto (\sin x)/2 + 0.5$ which is continuous. Hence the image of a connected set is connected, and $[0, 1]$ is connected.