1 Fri 10/9

From last lecture, if $S \subset M$ is connected then $S \subset T \subset \overline{S} \implies T$ is connected.

Example 1.1

Let $M = \mathbb{R}^2$ and $S = \{(x, \sin(1/x) \mid x \in (0,1]\}$, the topologist's sine curve itself. Recall that $\overline{S} = S \cup \{0\} \times [-1,1]$[see HW for justification], and S is connected because it is a continuous image from $(0,1] \to \mathbb{R}^2$. Then it immediately follows that

$$S \cup \{(0,x)\} \text{ for some } x \in [-1,1]$$

is connected since this set is a supset of S and a subset of $\overline{S}$.

Definition 1: Path

Let (M,d) be a metric space. A **path** in M from $p \in M$ to $q \in M$ is a continuous function $\gamma : [0,1] \to M$ with $\gamma(p) = p$ and $\gamma(1) = q$. "The path can be drawn with a pen and the dots on the path cannot teleport."

Definition 2: Path-connectedness

M is **path-connected** if any two points in M can be joined by a path.

Proposition 3

Path connectedness is stronger than connectedness: the former implies the latter.

Proof

Look at the contrapositive. If M is disconnected, then there exists a separation $M = A \sqcup B$ where A and B are both propen clopen. Pick $a \in A$ and $b \in B$. If there exists a continuous function $f : [0,1] \to M$ creating this path, then we know $0 = f^{-1}(a) \in f^{-1}(A)$ and $1 = f^{-1}(b) \in f^{-1}(B)$, but clearly $f^{-1}(A)$ and $f^{-1}(B)$ are proper clopen subsets of $[0,1]$. (They are disjoint.) Hence $[0,1]$ is disconenected, but this is absurd. □

Remark

On the other hand, connectedness does not necessarily imply path-connectedness. Consider the closed topologist's sine curve [see HW8].

Theorem 4

Any open subset of $\mathbb{R}$ is a countable union of disjoint intervals (possibly infinite or half infinite). Examples: $(0,1)$, or $(0,1) \cup (1,2)$, and so on.

Covering Compactness

Definition 5

A **covering** of a metric space M is a collection $\mathcal{U}$ of subsets $(V_\alpha)_{\alpha \in I}$ (I being some indexing set) such that $\bigcup_{\alpha \in I} V_\alpha = M$ (note that there is no supset of M). Likewise, a **covering** of a subset $A \subset M$ is a collection of subsets $(V_\alpha)_{\alpha \in I}$ of M such that $\bigcup_{\alpha \in I} V_\alpha \supset A$.

Definition 6

An **open covering** is a covering such hat each of the subsets V_α is open.

Definition 7

A **subcovering** is a covering including a subset of the collection of subsets in the original covering.

Definition 8

A metric space M is called **covering compact** if any open open covering *has* (can be reduced to) a finite subcovering.

Theorem 9

(To be proven later) Covering compactness is equivalent to sequential compactness.

Example 1.2

(1) $(-\infty, 1) \cup (-2, 2) \cup (-1, \infty)$ is an open covering of $\mathbb{R}$. A subcovering is $(-\infty, 1) \cup (-1, \infty)$.

(2) $\mathbb{R}$ can also be covered by

$$\{(n, n+2) \mid n \in \mathbb{Z}\} = \cdots \cup (0, 2) \cup (1, 3) \cup (2, 4) \cup \ldots$$

is an infinite open covering of $\mathbb{R}$. Notice that it does not have any finite subcovering! For example if we take away $(1, 3)$ then 2 is no longer in the covering. Hence we've found a (one suffices) covering of $\mathbb{R}$ that has no finite subcovering. Hence $\mathbb{R}$ **is not covering compact.**

(3) $\mathbb{Z}$ is again not covering compact because

$$\mathbb{Z} = \cdots \cup \{-1\} \cup \{0\} \cup \{1\} \cup \ldots$$

is an infinite open covering that has no finite subcovering.

Covering compact implies sequentially compact

Assume M is covering compact. Suppose by contradiction that it is not sequentially compact. Then we can find a sequence $(p_n) \in M$ without any convergent subsequence. Then for any $m \in M$, there exists $\epsilon > 0$ satisfying that only finitely many terms of (p_n) are inside $B_\epsilon(m)$. Now consider

$$\bigcup_{m \in M} B_\epsilon(p).$$

By covering compactness, this covering reduces to a finite covering with each subset containing finitely many points. However the sequence is infinite. By pigeonhole, a contradiction has to appear. Hence M must be sequentially compact. $\square$

2 Mon 10/12 Nested Sets & Cantor Sets

Cantor Intersection Theorem

Given a decreasing nested sequence $(M \supset C_1 \supset C_2 \supset \dots)$ of compact, nonempty subsets of a metric space (M, d), their intersection is compact and nonempty. Moreover, if the diameter $\to 0$, then the intersection is a single point.

Definition 11

The **diameter** of a set S is

$$\operatorname{diam}(A) := \sup_{x \in S, y \in S} d(x, y).$$

Remark

This can be blatantly false if the C's are not compact:

$$\bigcap_{i=1}^{\infty} (0, 1/i) = \varnothing.$$

Proof

First notice that $\bigcap C_i$ is an arbitrary union of closed sets and is therefore closed. It is also a subset of all the C_i's. Therefore $\bigcap C_i$ is compact. (Note that $\varnothing$ is also compact. Now we show that this intersection is nonempty.)

Since each C_n is nonempty, we can construct (x_n), a sequence with $x_i \in C_i$. Then clearly $(x_n) \in C_1$ and the compactness implies that it has a subsequence converging to $x \in C_1$. Then, there must also be a sub-subsequence of (x_n) that converges in C_2 since all but first terms of the original subsequence of (x_n) are guaranteed to lie in C_2, which guarantees the existence of a convergent sub-subsequence in C_2. Since limits are unique we know $x \in C_2$. Likewise, $x \in C_2$ and so on. Therefore $x \in \bigcap C_i$ and we've shown it is indeed nonempty.

For the diameter part, this is immediate by the non-emptyness of $\bigcap C_i$ and the fact that its diameter is 0, because otherwise the distance between two distinct points is positive, contradicting diameter $\to 0$. Phrased formally, suppose $\lim_{i \to \infty} \operatorname{diam}(C_i) = 0$, and $p, q \in \bigcap C_i$, if $p \neq q$, then we can pick $N \in \mathbb{N}$ such that

$\text{diam}(C_N) < d(p,q)$. Then

$$d(p,q) \leqslant \text{diam}(C_N) < d(p,q),$$

contradiction, since $p, q \in C_N$ by definition. Therefore $p = q$. □

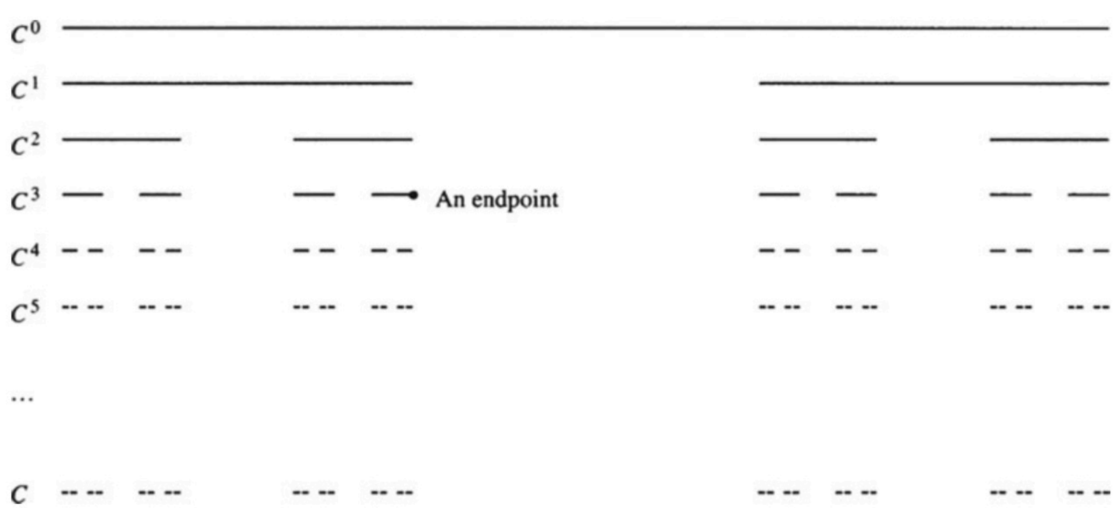
Definition 12: Standard middle thirds Cantor Set

The standard (middle-thirds) Cantor set is the intersection of the nested sequence

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C^n,$$

where $C^0 = [0,1]$ and C^{n+1} is obtained from removing the middle thirds from each interval in C^n. For example,

$$C^0 = [0,1]$$
$$C^1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$
$$C^2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$
$$\dots$$



Lemma 2.1

$\mathcal{C}$ is nonempty: think about the fact that each C^i is compact and nonempty. Or notice that $0, 1 \in \mathcal{C}$.

Theorem 13

$\mathcal{C}$ is uncountable.

Proof

We introduce a coordinate system: let $C^1 = C_0 \cup C_2$ where $C_0 = [0, 1/3]$ and $C_2 = [2/3, 1]$. Then let the four closed intervals of C^2 be $C_{00}, C_{02}, C_{20}, C_{22}$, from left to right, respectively. Keep doing this. For example C_{220} is an interval in C^3 and also the left subinterval of C_{22}, i.e., $[0, 1/27]$. [To be finished next lecture.] $\square$

3 Tue 10/13 Discussion

Recall the following concepts:

(1) Path-connected: a metric space M is said to be **path-connected** if for all $p, q \in M$ (different), there exists a continuous $f : [0, 1] \to M$ with $f(0) = p, f(1) = q$.

(2) Path-connectedness, just like compactness and connectedness, is a topological property. If $f : M \to N$ is continuous and M is path-connected, then $f(M)$ is also path-connected.

Proof

Pick $p, q \in f(M)$, consider the points $f^{-1}(p), f^{-1}(q) \in M$. By the path-connectedness of M we have a continuous $\gamma : [0, 1] \to M$ with $\gamma(0) = f^{-1}(p), \gamma(1) = f^{-1}(q)$.Therefore the composite $f \circ \gamma : [0, 1] \to N$ satisfies $f\gamma(0) = p, f\gamma(1) = q$, and of course the composite of continuous functions is still continuous. $\square$

(3) Path-connectedness implies connectedness but not necessarily the converse. Consider the closed topologist's sine curve.

(4) In $\mathbb{R}$, TFAE:

(I) I is path connected;

(II) I is connected;

(III) I is an interval. Definition of an interval: for all $a, b \in I$ with $a < b$, if $c \in (a, b)$ then $c \in I$.

(I) $\Longrightarrow$ (II) is trivial.

For (II) $\implies$ (III), take the contrapositive. Suppose I is not an interval, then there exist $a, b \in I$ with $a < b$ such that there exists some $c \in (a, b)$ with $c \notin I$.Let $A = \{x \in I \mid x < c\}$ and $B = \{x \in I \mid x > c\}$. Note that A and B are disjoint and nonempty. Now notice that $A = (-\infty, c) \cup I$ which, by the inheritance principle, implies A is open. Likewise for B. Then taking the complements implies that A and B are clopen. Hence $A \sqcup B$ is a nontrivial separation of I and I is disconnected.

For (III) $\implies$ (II), if I is an interval, then for all $p, q \in I$, the function $f : [0, 1] \to I$ defined by

$$x \mapsto (1 - x)p + xq$$

shows that I is path connected.

(5) A subset $E \subset M$ is said to be **covering compact** if for any open covering of E has a finite subcovering. For example, $(0, 1]$ is not compact because the covering

$$\bigcup_{i=1}^{\infty} (\frac{1}{i}, 2)$$

has no finite subcovering but it indeed covers $(0, 1]$.

Worksheet # 9

Problem 1

Given (M, d) a metric space, and fix some $x_0 \in M$.

(1) Define a function $f : M \to \mathbb{R}$ by $f(y) = d(x_0, y)$. Show that f is continuous.

(2) Let $A \subset M$ be a compact subset and $x_0 \notin A$. Show that there exists $z \in A$ such that $d(x_0, z) \leqslant d(x_0, y)$ for all $y \in A$.

Solution

(1) Use the definition of sequential continuity: suppose $(y_n) \in M$ converges to y, then

$$d(x_0, y_n) \leqslant d(x_0, y) + d(y_n, y) \implies d(x_0, y) \leqslant d(x_0, y_n) + d(y_n, y)$$
$$d(x_0, y) \leqslant d(x_0, y_n) + d(y_n, y) \implies d(x_0, y) - d(x_0, y_n) \leqslant d(y_n, y)$$

Then we see indeed $(x_0, (y_n)) \to (x_0, y)$. Hence f is continuous.

(2) Let $h : A \to \mathbb{R}$ with $y \mapsto d(x_0, y)$ which is continuous. This is f restricted on A. Since the continuous image of a compact set is compact and therefore attains maximum and minimum, we know there exists $z \in A$ with $h(z) \leqslant h(y)$ for all $y \in A$.

Problem 2

Prove that every convex subset $E \subset \mathbb{R}^m$ is path connected.

Solution

Immediate from the definition of convexness. Suppose $p, q \in \mathbb{R}^m$ then

$$\lambda p + (1 - \lambda)p \in E.$$

Now if we define $f : [0, 1] \to E$ by

$$\lambda \mapsto \lambda p + (1 - \lambda)p$$

then this continuous function shows that E is path connected.

Problem 3

If $A \subset M$ is path connected, is $\overline{A}$ necessarily path connected?

Solution

No. Consider the topologist's sine curve and its closure.

Problem 4

Given a compact metric space (M, d), and f a function from M to N such that

$$d(f(x), f(y)) < d(x, y)$$

if $x \neq y$, prove that f has a unique fixed point in M, i.e., there exists a unique $x_0 \in M$ such that $f(x_0) = x_0$.

Solution

First we show the existence: define $h : M \to \mathbb{R}$ by $x \mapsto d(x, f(x))$. Claim: h is continuous. Suppose $(x_n) \to x$, then

$$\begin{aligned} f(x_n, f(x_n)) &\leqslant d(x_n, x) + d(x, f(x)) + d(f(x), f(x_n)) \\ d(x_n, f(x_n)) - d(x, f(x)) &\leqslant d(x_n, x) + d(f(x), f(x_n)) \\ d(x, f(x)) &\leqslant d(x, x_n) + d(x_n, f(x_n)) + d(f(x_n), f(x)) \\ d(x, f(x)) - d(x_n, f(x_n)) &\leqslant d)x_n, x) + d(f(x_n), f(x)) \end{aligned}$$

As $n \to \infty$, we have

$$|d(x, f(x)) - d(x_n, f(x_n))| \leqslant d(x_n, x) + d(f(x_n), f(x)).$$

Hence f is continuous over a compact set M. Hence there exists $x_0 \in M$ satisfying $h(x_0) = 0$. Otherwise $h(x_0) > 0$ and so $d(x_0, f(x_0)) > d(f(x_0), f(f(x_0)) = h(f(x_0))$, and this shows x_0 is no longer the minimum.

Now for the uniqueness, assume there are two x_0, y_0 with $d(x_0, y_0) = d(f(x_0), f(y_0))$. This contradicts the very assumption that LHS > RHS.

4 Wed 10/14

Continuing on proving that $\mathcal{C}$ is uncountable:

Proof

(We introduce a coordinate system: let $C^1 = C_0 \cup C_2$ where $C_0 = [0, 1/3]$ and $C_2 = [2/3, 1]$. Then let the four closed intervals of C^2 be $C_{00}, C_{02}, C_{20}, C_{22}$, from left to right, respectively. Keep doing this. For example C_{220} is an interval in C^3 and also the left subinterval of C_{22}, i.e., $[0, 1/27]$.)

Define $C_{i_1 i_2 i_3 \dots}$ as the intersection $C_{i_1} \cap C_{i_1 i_2} \cap C_{i_1 i_2 i_3} \cap \dots$. This intersection is clearly in $\mathcal{C}$. Since each subset is compact [closed and bounded in $\mathbb{R}$], by the Cantor Intersection Theorem their intersection will be a single point. Therefore we see that there is a bijection between points $p \in \mathcal{C}$ and the set of infinite interceptions $C_{i_1 i_2 i_3 \dots}$. Hence there exists a bijection between this set and $\mathcal{C}$. Recall $\sum_2$, the set of infinite words from an alphabet of 2 letters, which is uncountable. Hence $\mathcal{C}$ is uncountable. □

Definition 15

Let (M, d) be a metric space and let $S \subset M$ be a set. A **cluster point** of $p \in S$ is a point such that every open neighborhood of p in M contains infinitely many points of M.

Definition 15

If p is not a clutter point, it is called an **isolated point**. That is, if there exists some neighborhood that contains only p.

Remark

To show that p is a cluster point, showing that there exists *one* point in any arbitrary neighborhood of p suffices. Seeting ϵ and then setting smaller neighborhoods of radii $\epsilon/2, \epsilon/3, \dots$ guarantees the existence of a sequence that converges to p.

Definition 16

A metric space (M, d) is said to be **perfect** if every point in M is a cluster point.

Theorem 17

$\mathcal{C}$ is perfect.

Proof

Pick $p \in \mathcal{C}$ and $\epsilon > 0$. We can *always* find a sufficiently large $n \in \mathbb{N}$ satisfying $1/3^n < \epsilon$. We do this because the interval lengths of C^n decreases by a factor of 3. This way both endpoints of the invertal of length $1/3^n$ containing p is in the ϵ-neighborhood. Therefore the neighborhood always contains some other points and p is a clustering point. □

Theorem 18

Any nonempty, perfect, complete metric space M is uncountable. [To be continued next time; Pugh p.94]

5 Fri 10/16 More on Cantor Sets

Back to the proof of theorem above: nonempty, perfect, complete implies uncountable. Notice that if M is not complete then it can be nonempty, perfect, AND countbable: think of $\mathbb{Q}$.

Proof

Suppose, by contradiction, that (M, d) is nonempty, perfect, complete, but countable (countably infinite). Then we can enumerate the points as $M = \{x_1, x_2, \dots\}$. Our goal is to find a point in M that is not in the enumeration so that we derive a contradiction.

First pick $y_1 \neq x_1 \in M$ (also in the enumeration above). Since $y_1 \neq x_1$ we can "surround y_1 with some ball whose closure excludes x_1": we can pick $1 > r_1 > 0$ satisfying $x_1 \notin \overline{B_{r_1}(y_1)}$. Since y_1 is [must be] a cluster point, there are infinitely many points of M in $B_{r_1}(y_1)$ (or its closure). Then pick $y_2 \in B_{r_1}(y_1) \subset M$ such that $1/2 > y_2 \neq x_2$ and pick $r_2 > 0$ small enough such that $x_2 \notin \overline{B_{r_2}(y_2)}$ [which is possible since $B_{r_1}(y_1)$ is open]. Then we know continue such construction since each point in M is a cluster point. Then the sequence of balls $B_{r_1}(y_1), B_{r_2}(y_2), \dots$ are a nested sequence of open balls. On the other hand, the centers, $y_1, y_2, \dots$, form a Cauchy sequence. By completeness of M this sequence converges to $y \in M$, say.

On one hand, any sequence of $(y_1, y_2, \dots)$ starting from the i'th term is enclosed entirely in the closed ball $\overline{B_{r_i}(y_i)}$. Hence the limit y must lie in all the balls.

On the other hand, $y \neq x_1$ since x_1 is excluded by the first ball; it's not x_2 since it is excluded by the second ball. More generally, it is not any x_n from the enumeration $\{x_1, x_2, \dots\}$ of M. Contradiction. Hence we cannot enumerate points in M. □

Definition 19

A subset S of a metric space (M, d) is **dense in** M if $\overline{S} = M$.

Example 5.1

$\mathbb{Q}$ is dense in $\mathbb{R}$ since $\overline{\mathbb{Q}} = \mathbb{R}$, while $\mathbb{Z}$ is not since $\overline{\mathbb{Z}} = \mathbb{Z} \neq \mathbb{R}$.

Definition 20

If S is a subset of M and U is an open subset of M, we say S is **dense in** U if $\overline{S \cap U} \supset U$.

Definition 21

A subset $S \subset M$ is **nowhere dense** if it is not in any (nonempty) open subset of M.

Theorem 22

$\mathcal{C}$ contains no intervals (a, b). Moreover, $\mathcal{C}$ is nowhere dense in $\mathbb{R}$.

Proof

Suppose $(a, b) \in \mathcal{C}$. Pick $n \in \mathbb{N}$ such that $1/3^n < b - a$. If $(a, b) \subset \mathcal{C}$, then $(a, b) \subset C^n$ which is clearly a contradiction since each interval in C^n is smaller than (a, b) itself.

Now imagine that we can find an open subset $U \in \mathbb{R}$ such that $\overline{\mathcal{C} \cap U} = U$. Then we can find an interval $(a, b) \in U$, and so

$$(a, b) \subset U \subset \overline{\mathcal{C} \cap U} \subset \overline{\mathcal{C}} = \mathcal{C},$$

contradiction. □

Definition 23

A subset $S \subset \mathbb{R}$ is a **null set** (also called a **zero set**) if, for any $\epsilon > 0$, the subset S can be expressed as a union of countably intervals (a_i, b_i) such that

$$\sum_{i=1}^{\infty}(b_i - a_i) < \epsilon.$$

(In other words, this set needs to have zero outer measure.)

6 Mon 10/19

Definition 24

A metric space M is **totally disconnected** if for any $\epsilon > 0$ and any $p \in M$, we can find a clopen subset of M containing p and contained in $B_\epsilon(p)$.

Theorem 25

$\mathcal{C}$ is totally disconnected.

Proof

Take $p \in \mathcal{C}$ and any $\epsilon > 0$. We can find an interval I containing p and also contained in $(p - \epsilon, p + \epsilon)$. Now consider $U = \mathcal{C} \cap I$ — we want to show this is a clopen subset of $\mathcal{C}$.

First thing, I is a closed interval so it's closed in $\mathbb{R}$. Then the inheritance principle suggests that $U = (\mathcal{C} \cap I) \subset \mathcal{C}$ is the intersection of $\mathcal{C}$ and a closed subset of $\mathbb{R}$ and is therefore closed.

On the other hand, consider the complement of $U = \mathcal{C} \setminus U = \mathcal{C} \cap (C^n \setminus I)$, and $C^n \setminus I$ is the union of $3^n - 1$ closed intervals. Therefore by inheritance principle again, the complement of U is closed and U is open. Therefore U is clopen. $\square$

In summary, we have shown:

Theorem 26

$\mathcal{C}$ is compact, nonempty, perfect, totally disconnected, uncountable, and has measure 0.

Definition 27

Any metric space satisfying the first 4 conditions, i.e., compact, nonempty, perfect, and totally disconnected, is called a **Cantor space**.

Moore-Kline Theorem

Any two cantor spaces are homeomorphic.

Completion

Theorem 29

For every metric space (M, d), there's a complete metric $\hat{M}, \hat{d}$ such that M is a dense subset of $\hat{M}$ and d is the induced metric. Moreover, $(\hat{M}, \hat{d})$ is unique up to isometry.

Sketch of proof

Let $\hat{M}$ be the set of *all* Cauchy sequences in M. We define $\sim$ to be an equivalent relation such that, for Cauchy sequences $(p_n), (q_n)$, $(p_n) \sim (q_n)$ if $\lim_{n\to\infty} d(p_n, q_n) = 0$. Then define

$$\hat{d}((p_n), (q_n)) := \lim_{n\to\infty} d(p_n, q_n).$$

Claim: $\hat{d}$ is well-defined, i.e., independent of the preventative of equivalence class (many Cauchy sequences converge to the same limit and we can pick co-Cauchy sequences freely). Also $\hat{M}$ is complete and $M \subset \hat{M}$. □

Cantor Surjection Theorem

Give any compact nonempty metric space (M, d), there exists a continuous surjection $\mathcal{C} \to M$. This leads to **space filling curves**:

Theorem 31

The **Peano curve** can be written as an image of a surjective continuous map $\gamma : [0, 1] \to B^2$. Furthermore, by the Cantor surjection theorem, we can find a continuous surjective $\sigma : \mathcal{C} \to B^2$. See Pugh pg.112.

7 Tue 10/20 Midterm II Review

About Continuous Functions

Suppose $f : M \to N$ is continuous. The following are true:

(1) The preimage of any open $S \subset N$ under f is open.

(2) The preimage of any closed $S \subset N$ under f is closed.

(3) The image of any compact/connected/path-connected set $T \subset M$ is compact/connected/path-connected, i.e., these properties are preserved.

The following are NOT necessarily true:

(1) The continuous image of open set is open: consider $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto \sin x$. Then $f((-\infty, +\infty)) = [-1, 1]$, open to non-open.

(2) The continuous image of closed set is closed: consider the Signoid function $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto e^x/(1 + e^x)$. Then $f([0, +\infty)) = (0.5, 1)$, closed to non-closed.

(3) The preimage of a connected/path-connected set $S \subset N$ under f is connected/path-connected. Consider $f(x) : x \mapsto x^2$ and $S = [1, 4]$. Then $f^{-1}([1, 4]) = [-2, -1] \cup [1, 2]$, connected image with disconnected preimage.

(4) The continuous image of a bounded set is bounded / the preimage of complete set is complete. Consider $\tan x$ where $f((-\pi/2, \pi/2)) = \mathbb{R}$.

(5) The continuous image of a complete set is complete / the preimage of a bounded set is bounded. Similar to above, consider $\arctan x$ where $f(\mathbb{R}) = (-\pi/2, \pi/2)$.

Some Classic Homeomorphisms and Non-Homeomorphisms

Classic homeomorphisms:

(1) $(0, 1) \cong \mathbb{R} : f(x) = \tan(\pi(x - 1/2))$.

(2) $S^1 \setminus \{(0, 1)\} \cong \mathbb{R}$: extend the line between $(0, 1)$ and any point on $S^1 \setminus \{(0, 1)\}$ and find its intersection with x-axis.

(3) $D_1 \cong \mathbb{R}^2$.

$$f(r, \theta) = (\tan(\frac{\pi r}{2}) \cos\theta, \tan(\frac{\pi r}{2}) \sin\theta).$$

Non-homeomorphisms:

(1) $S^1 \ncong [0, 2\pi)$. The function $(\cos\theta, \sin\theta) \mapsto \theta$ is not bicontinuous. Think of the $2\pi = 0$ issue.

(2) $S^1 \ncong \mathbb{R}^n$. For $n = 1$, remove one point from both sides. S^1 minus one point is connected but R minus one point is not. For $n \geqslant 2$, remove two points on both sides. S^2 minus two points is disconnected but $\mathbb{R}^n$ is still connected.

From Topic List

(1) Metric spaces (M, d), especially $(\mathbb{R}^n, d_{\text{st}})$ and the discrete metric space (M, d^*) which is very useful for constructing counterexamples.

  (1) If (M, d) is a finite metric space, it is disconnected — actually totally disconnected. Let $r < \inf_{x,y \in M} d_M(x, y)$ then each $p \in M$ is enclosed by some clopen neighborhood that only contains itself.

  (2) A finite metric space is also compact. By pigeonhole, if (p_n) is a sequence in M that has infinitely many terms, there's at least one $m \in M$ that appears infinitely many times in the sequence. Then the subsequence $(m, m, \dots) \to m$ and hence M has a convergent subsequence. So it's compact.

(2) To show one set is closed: show it contains all its limit points or show its complement is open. To show one set is open: show each point in it has some neighborhood contained in the set, or show its complement is closed.

(3) Definitions of continuity: let $f : M \to N$ be a function. TFAE

  (1) f is continuous if and only if it preserves sequential convergence.

  (2) f is continuous if and only if it satisfies the $\epsilon - \delta$ condition.

  (3) f is continuous if and only if the preimage of every open set $S \subset N$ under f is open (in M).

  (4) f is continuous if and only if the preimage of every closed set $S \subset N$ under f is closed (in M).

(4) Some topology $(M, \mathcal{T})$.

  (1) Finite union or arbitrary intersection of closed sets is closed. (Think of $\bigcup\{x\}$ with $x \in (0, 1)$.)

  (2) Finite intersection or arbitrary union of open sets is open. (Think of $\bigcap(1 - 1/n, 2 + 2/n), n \in \mathbb{N}$.)

(5) Given a set S we have $\text{int}(S) \subset S \subset \overline{S}$. The former is open and the latter closed.

(6) Product metrics: given M and N, $d_{\text{sum}}, d_{\text{Euclidean}}, d_{max}$ all work as metric for $M \times N$.

(7) Compact implies closed and bonded. **Closed & bounded do not necessarily imply compact.** Counterexample includes $\mathbb{N}$ with discrete metric. This statement is, however, true in $\mathbb{R}^n$ as guaranteed by Heine-Borel theorem.

(8) In $\mathbb{R}^n$, connected $\iff$ path-connected $\iff$ the set is an interval.

(9) Cluster point: $p \in E$ is a cluster point if (TFAE)

  (1) there exist infinitely many points within *any* neighborhood of p.

  (2) there exist one point other than p within *any* neighborhood of p.

  (3) there exist at least two points in each neighborhood of p.

  (4) there exists a sequence (not $p, p, \dots$) converging to p.

(10) Isolated point: not a cluster point. Exists some neighborhood within which there's no other point.

(11) A perfect metric space is one such that every point in it is a cluster point.

(12) Cantor set $\mathcal{C}$:

$$\mathcal{C} = \{\sum_{i=1}^{\infty} \frac{a_n}{3^n}, a_n \in \{0,2\}\}.$$

  (1) Anything set, like $\mathcal{C}$, that is compact, nonempty, perfect, and totally disconnected, is called a Cantor space.

  (2) $\mathcal{C}$ is uncountable. It's also a null set.

7.1 Worksheet # 10

Problem 5

(1) $S^1 \ncong [0,1)$ because if we remove one point from both sides, one become connected and the other disconnected.

(2) if $f : M \implies N$ is injective, then $f : M \to f(M)$ is bijective. True. It's injective obviously and it's surjective because everything in $f(M)$ always gets mapped to.

(3) Every subset of the discrete metric space is clopen. True.

(4) Any nonempty subset of $\mathbb{R}$ contains at least one nonempty compact subset. True. Singletons are compact in $\mathbb{R}$.

(5) In any metric space (M,d), the empty set and M are compact. False. M itself can be not compact. For example $\mathbb{R}$.

(6) If $S \subset N \subset M$ and S is open in M, then S is open in N. True by inheritance principle: $S \cap N = S$, open in N.

(7) If $A, B \subset M$. If A, B are connected then so is $A \cup B$. False. Disjoint intervals in $\mathbb{R}$ for counterexample.

(8) $\mathcal{C}$ does not contain any irrational since the endpoints of each interval is rational. False. $\mathcal{C}$ is uncountable but $\mathbb{R}$ is. It must contain some irrationals.

(9) If M is not path-connected then it's disconnected. False. Counterexample: closed topologist's sine curve.

Problem 6

Suppose (M, d) is a metric space. Let $p \in M$ and $\delta > 0$. Show that $W_p(\delta) := \{q \in M \mid d(p, q) > \delta\}$ is open in M.

Solution

The complement of this set is $\overline{B_\delta(p)}$ which is closed. Hence $W_p(\delta)$ is open. More rigorously, we want to show that for each limit point q of the complement, it's also in the complement. By the existence of a convergent sequence $(q_n) \to q$ we can find some n satisfying $d_(q_n, q) < \epsilon$ and for all later terms. Then

$$d(p, q) \leqslant d(p, q_n) + d(q_n, q) \leqslant \epsilon + \delta.$$

Problem 7

Show that every connected metric space (M, d) with at least two points is uncountable.

Solution

Suppose M is countable and has at least two points. Let $x_0 \in M$. Define $f : M \to \mathbb{R}$ by $y \mapsto d(x_0, y)$. Since f is continuous, we must have $f(M)$ a connected subset of $\mathbb{R}$, i.e., an interval. Since an interval is uncountable, so is M.