1 Mon 9/14

Metric Spaces

Definition 1

A **metric space** is a pair (M, d), where M is a set and d is a function $M \times M \to \mathbb{R} \geqslant 0$ that satisfies three properties:

(1) positive definiteness: $d(x, y) \geqslant 0$ and $d(x, y) = 0$ if and only if $x = y$.

(2) symmetry: $d(x, y) = d(y, x)$.

(3) triangle inequality: $d(x, z) \leqslant d(x, y) + d(y, z)$.

Example 1.1

$\mathbb{R}^n$ with $d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$, our common sense definition of the "distance" between two points in $\mathbb{R}^n$.

Example 1.2

$\mathbb{R}^n$ with $d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \|\mathbf{x} - \mathbf{y}\|$.

Definition 2

In general, the L^p**-norm**

$$\|(x_1, \ldots, x_n)\|_{L^p} = \sqrt[p]{|x_1|^p + |x_2|^p + \cdots + |x_n|^p}$$

and the ∞**-norm** is

$$L^\infty : |(x_1, x_2, \ldots, x_n)| = \max_{i=1,\ldots,n} |x_i|$$

Remark

While inner product induces norm, norm also induces metric space.

Example 1.3

Another metric that has nothing to do with norm: the **discrete metric**:

Let M be a set and define the discrete metric

$$d(x,y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}$$

Easy to verify positive definiteness, symmetry, and triangle inequality.

Example 1.4

$(\mathbb{R}^n, d_{\text{discrete}})$ is a metric space.

Remark

If (M,d) is a metric space, then for any subset of $N \subset M$, then (N,d) is also a metric space if we restrict d to N. In this case we call d the induced / inherited metric.

Remark

For convenience we denote (M,d) as M unless there's need to state what metric d is.

Example 1.5

$\mathcal{S} = \{(x,y) \in \mathbb{R}^2 \mid y = x^3 + 1\}$ with the induced metric from d_{st} (the standard metric) is a metric space.

Example 1.6

A more interesting metric space:

Consider $\mathcal{S} = \bigcup_{n=1}^{\infty} \{(x,y) \in \mathbb{R}^2 \mid d_{\text{st}}(x,y),(1/n,0) = 1/n\}$. A bunch of circles passing through the origin, centered on x-axis, with centers closer and closer to the origin and decreasing radius. This set forms a **Hawaiian Earring**.

Sequences

Definition 3

If (M, d) is a metric space, then a **converge** (p_n) is a list of points $p_1, p_2, \ldots$ with $p_i \in M$.

Definition 4

A sequence (p_n) converges in a metric space (M, d) **converges to a limit** $p \in M$ if

$$\forall \epsilon > 0, \exists\, N \in \mathbb{N} \text{ such that } n \geqslant N \implies d(p_n, p) < \epsilon.$$

Definition 5

If $(p_n) = p_1, p_2, \ldots$ is a sequence in (M, d), then a **subsequence** (p_{n_i}) is any sequence of the form $p_{n_1}, p_{n_2}, \ldots$ with $n_j > n_k$ for any $j > k$.

Example 1.7

Define (p_n) by $\frac{1}{2}, \frac{1}{3}, \ldots$. Once subsequence is $1, \frac{1}{5}, \frac{1}{10}, \ldots$.

Theorem 6

If (p_n), a sequence in (M, d), converges to $p \in M$, then any subsequence also converges to p.

Problem 1

If a sequence (p_n) has a convergent subsequence, does (p_n) necessarily converge?

Solution

No! Consider the sequence (p_n) defined by

$$p_i = \begin{cases} 1 \text{ if } i \text{ is even} \\ \frac{1}{i} \text{ if } i \text{ is odd} \end{cases}$$

Clearly the sequence has a subsequence $1, \frac{1}{3}, \frac{1}{5}, \dots$ that converges to 0, but the entire sequence does not converge to anything.

2 Tue 9/15 Discussion: Review for Test

Some notations:

(1) Let A, B be sets. We say $A \sim B$ if there exists $f : A \to B$, a bijection.

(2) We can check that $\sim$ defines an equivalence relation:

- (I) Reflexive because $A \sim A$ due to the identity map.
- (II) Symmetric because if $A \sim B$ by $f : A \to B$ then $B \sim A$ by f^{-1}.
- (III) Transitive because if $A \sim B$ by $f : A \to B$ and $B \sim C$ by $g : B \to C$ then $A \sim C$ by $g \circ f : A \to C$.

(3) A set $\mathcal{S}$ is called:

- (I) finite if there exists an $n \in \mathbb{N}$ such that $\mathcal{S} \sim \{1, 2, \dots, n\}$.
- (II) infinite if it is not finite.
- (III) denumerable if $\mathcal{S} \sim \mathbb{N}$.
- (IV) countable if finite or denumerable.
- (V) uncountable if not countable.

(4) $\mathbb{R}$ is uncountable: *Cantor's Diagonalization.*

(5) Some corollaries:

- (I) Any intervals of form $(a, b), [a, b], (a, b], [a, b)$ are uncountable. Consider $\tan^{-1}$: a bijection from $\mathbb{R}$ onto $(-\pi/2, \pi/2)$.
- (II) $\mathbb{Q}$ is countable: it is equivalent to $\mathbb{Z} \times \mathbb{N} \sim \mathbb{N} \times \mathbb{N} \sim \mathbb{N}$.
- (III) $\mathbb{R} \setminus \mathbb{Q}$ is uncountable: if not, then $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$, the union of two countable set, is countable.

Example 2.1

Suppose $\mathcal{S}$ is countable and $E \subset \mathcal{S}$, show that E is also countable.

Proof.

Case 1 : $\mathcal{S}$ is finite. The result is immediate since E is also finite.

Case 2 : $\mathcal{S}$ is denumerable. If $\mathcal{E}$ is finite then the result immediately follows.

Case 3 : $\mathcal{S}$ is denumerable and E is not finite. Since $\mathcal{S}$ is denumerable, let a_n be an enumeration of $\mathcal{S}$. Define $f : \mathbb{N} \to E$ by $n \mapsto$ the n^{th} element in $\mathcal{S}$ that is in E. Clearly f is injective since $n \neq m \implies f(n) \neq f(m)$. To show f is surjective, take any $y \in E$. Then $y = a_i$ for some $i \in \mathbb{N}$. Since a_i will appear somewhere in the enumeration a_n, there will exist some $j \in \mathbb{N}$ such that $f(j) = a_i$ by the construction of f. Hence f is also surjective and therefore objective. $\mathbb{N} \sim E$, and E is countable. ☐

Remark

Any infinite subset of a denumerable set is denumerable.

Example 2.2

If disjoint A and B are countable, then $A \sqcup B$ is countable.

Proof.

Trivial when both sets are finite.

When one is finite and the other denumerable, WLOG assume A is finite. Then there exists $n \in \mathbb{N}$ such that $A \sim \{1, 2, \ldots, n\}$ and there exists b_i, an enumeration of B. Then consider the function $f : \mathbb{N} \to A \cup B$ defined by

$$f(x) = \begin{cases} x, & \text{if } x \leqslant n \\ b_{x-n}, & \text{if } x > n \end{cases}$$

a bijection, and the enumeration looks like $a_1, \ldots, a_n, b_1, b_2, \ldots$

If both are denumerable then we can let a_i, b_j be enumerations of A and B. Define $g : \mathbb{N} \to A \cup B$ by

$$g(x) = \begin{cases} a_{x+1/2} & \text{if } x \text{ is odd} \\ a_{x/2} & \text{if } x \text{ is even} \end{cases}$$

another bijection, and the enumeration looks like $a_1, b_1, a_2, b_2, \ldots$ ☐

Remark

If the disjoint condition is absent, we can define $B' = B \setminus A$. Then $A \cup B = A \sqcup B'$ and the result follows.

Example 2.3

Let $A_1, A_2, \ldots, A_n$ be countable sets. Show that the union of these finitely many countable sets is countable.

Proof. We can prove by induction and the base case holds above. Then $(\bigcup_{i=1}^{n} A_i) \cup A_{n+1}$ is also countable. QED. □

Example 2.4

Given A_i countable, show that $\bigcup_{i=1}^{\infty} A_i$ is also countable.

Proof. Assume A_i's are disjoint, and assume they are all denumerable. (Doesn't differ too much when a set is finite. Consider $\{1,2,3\}$ as $\{1,2,3,3,3\ldots\}$ then later remove the redundancies.) Then let a_{in} be an enumeration of A_i. Then $f : \mathbb{N} \times \mathbb{N} \to \bigcup_{i=1}^{\infty} A_i$ defined by $(i, n) \mapsto a_{in}$ is a surjection. Since the union of denumerable sets cannot be finite, it must be infinite and thus denumerable. □

Worksheet 5 Problems

Problem 2

True or false. Justify.

(1) The set $A = \{x \in \mathbb{R} \mid 0 \leqslant x < 1\}$ is uncountable.

(2) A is uncountable and B is countable $\implies A \cap B$ is countable.

(3) Assume $\mathcal{S} \subset \mathbb{R}$ has a L.U.B., then $\sup(\mathcal{S})$ may or may not be in $\mathcal{S}$.

(4) For any index set I, if A_i is countable for any $i \in I$, then $\bigcup_{i \in I} A_i$ is countable.

(5) If there is a function $f : \mathbb{N} \to \mathcal{S}$ that is injective but not surjective, then $|\mathbb{N}| < |\mathcal{S}|$.

(6) Every subset of $\mathbb{R}$ has a L.U.B.

(7) On the set $\mathbb{R}$, the relation that $x \sim y$ defined by $|x - y| \leqslant 1$ is an equivalence relation.

(8) Let (a_n) be a sequence in $\mathbb{R}$. If (a_n) is convergent then (a_n/n) is also convergent.

(9) Let (a_n) be a sequence in $\mathbb{R}$. If (a_n) is divergent then (a_n/n) is also divergent.

Solution

(1) True; consider $\tan^{-1}$ with some coefficients.

(2) True; $A \cap B$ is a subset of B.

(3) True; consider $\{x \in \mathbb{R} \mid x \leqslant 2\}$ and $\{x \in \mathbb{R} : x < 2\}$.

(4) False: Let $I = \mathbb{R}$ and $A_i = \{i\}$. Then the union is just $\mathbb{R}$.

(5) False; f is not surjective doesn't mean there is no surjective $g : \mathbb{N} \to \mathcal{S}$. Note that $\leqslant$ is true by definition.

(6) Hell no. $\mathbb{Z} \subset \mathbb{R}$ for example.

(7) False; not transitive.

(8) True; and it's not hard to show that it converges to 0. If (a_n) is convergent then a_n is bounded. Therefore $|a_n| \leqslant M$ for some M and $|a_n/n| \leqslant \epsilon$ for some $n \geqslant M/\epsilon$.

(9) False: consider the sequence $0, 1, 0, 1, \ldots$.

Problem 3

Prove $\inf(\mathcal{S})$ where $\mathcal{S} = \{1/n \mid n \in \mathbb{N}\}$ is 0.

Proof. First note that 0 is a lower bound. Suppose $x > 0$ is also a lower bound, then if $n > 1/x > 0$ we have $1/n < x$. Hence x cannot be a lower bound. Therefore $\inf(\mathcal{S}) = 0$. ☐

Problem 4

Let $f : X \to Y$ be a function. Show that if $C, D \subset Y$ then $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

Solution

We need to show mutual inclusion.

For $\subset$, suppose $x \in f^{-1}(C \cup D)$, then either $f(x) \in C$ or $f(x) \in D$. Hence either $x \in f^{-1}(C)$ or $x \in f^{-1}(D)$. Hence $x \in f^{-1}(C) \cup f^{-1}(D)$.

For $\supset$, suppose $x \in f^{-1}(C) \cup f^{-1}(D)$, then either $x \in f^{-1}(C)$ or $x \in f^{-1}(D)$. Hence $f(x) \in C$ or $f(x) \in D$, and $x \in f^{-1}(C \cup D)$.

Problem 5

Let (a_n) be a convergent sequence and assume $\lim_{n\to\infty} a_n = A \neq 0$. Prove that

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{A}$$

Proof. Idea: we want to prove that given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geqslant N \implies |1/a_n - 1/A| < \epsilon$. Note that $|1/a_n - 1/A| = |(A - a_n)/(a_n A)|$.

Back to the proof: by convergence of a_n we have: given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geqslant N \implies |a_n - A| < \epsilon$. WLOG assume $A > 0$, then we can set $\epsilon < A/2$. Then there exists N_1 such that

$$\text{if } n \geqslant N_1,\ |a_n - A| < \frac{A}{2} \implies \frac{A}{2} < a_n < \frac{3A}{2}.$$

There also exists N_2 such that $n \geqslant N_2 \implies |a_n - A| < \epsilon/(A^2/2)$. Then if we let $n \geqslant \max\{N_1, N_2\}$ we have

$$|\frac{1}{a_n} - \frac{1}{A}| = |\frac{A - a_n}{a_n A}| < |\frac{\epsilon/(A^2/2)}{\frac{A}{2} \cdot A}| = \epsilon.$$

This concludes the proof of convergence. □

3 Fri 9/18

Recall from last class:

(1) A metric on a set M is a function $d : M \times M \to \mathbb{R}_{\geqslant 0}$ that is positive semidefinite symmetric, and has triangle inequality.

(2) Example of metric: standard distance between two points in $\mathbb{R}^n$, $d_{\text{st}}(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.

(3) Subset and inherited metric.

(4) Any finite subset of $\mathbb{R}^n$ with induced metric from the standard metric is a metric space.

Example 3.1

Consider the metric of a finite space: $M = \{a, b, c\}$ and the discrete metric is defined by

metric	a	b	c
a	0	1	1
b	1	0	1
c	1	1	0

whereas all the followings fail to be metrics (entry a, b corresponds to $d_M(a, b)$: the first is not symmetric, the second has $d_M(a, a) \neq 0$, and the first fails to satisfy triangle inequality with $d_M(a, c) > d_M(a, b) + d_M(b, c)$.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

However the following is a valid metric with entry a, b corresponding to $d_M(a, b)$.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Problem 6

Does there exist a subset of 3 points in $\mathbb{R}^1, \mathbb{R}^2$ such that the induced metric on that set is the discrete metric?

Solution

Not in $\mathbb{R}^1$ because $1 + 1 = 2$. However, this can be achieved in $\mathbb{R}^2$: consider the three vertices of an equilateral triangle.

Remark

Four points as the vertices of a unit tetrahedron in $\mathbb{R}^3$ would be the answer for a set of 4 points.

Definition 7

Let (M,d) and (M',d') be metric spaces. A map $f : M \to M'$ is **continuous at** $p \in M$ if $\forall \epsilon > 0$, there exists $\delta > 0$ such that if $x \in M$, then$d_M(x,p) < \delta \implies d_{M'}(f(x), f(p)) < \epsilon$. f is **continuous** if it is continuous at every $p \in M$.

Example 3.2

Suppose (M, d_m) and N, d_N are finite metric spaces. When is a function $f : M \to N$ continuous?

Solution

First observe that if f is a constant function (i.e., there exists $n \in N$ such that $f(m) = n$ for all $m \in M$). If this is the case, $d_N(f(m_1), f(m_2)) = 0$ for all $m_1, m_2 \in M$ and continuity follows.

If f is not a constant, we can still take δ sufficiently small that $\delta < d_M(m_1, m_2)$ for any $m_1, m_2 \in M$. Then $d_M(m_1, m_2) < \delta \implies m_1 = m_2 \implies d_N(f(m_1), f(m_2)) = 0 < \epsilon$.

Example 3.3

When is a function from $(\mathbb{R}^n, d_{\text{st}})$ to a finite metric continuous?

Solution

If and only if it's a constant function. [Why?]

Definition 8

A function $f : M \to N$ (shorthand notation for two metric spaces) is **sequentially continuous** if, for any sequence $(p_n) \in M$ that converges to p, then the corresponding sequence $(f(p_n)) \in N$ converges to $f(p)$.

Example 3.4

An example that doesn't preserve sequential convergence: consider the sequence $(1, \frac{1}{2}, \frac{1}{3}, \dots)$ and the function $f(x) = \frac{1}{x}$. The original sequence converges but the thing it gets mapped to is the arithmetic sequence $(1, 2, \dots)$.

Example 3.5

A more innocent-looking function: consider $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ defined by

$$f(x) = \begin{cases} x \text{ if } x > 0 \\ 17 \text{ if } x = 0 \end{cases}$$

If $(p_n) = 1/n$ and $(f(p_n)) = 1/n$, both converge to 0. However $0 \neq f(0)$.

Theorem 9

f is continuous if and only if it's sequentially continuous. [Will prove in next class]

Corollary 10

The composition of two continuous functions is also continuous.

Proof. Especially easy by using the property of sequential continuity. Define

$$(M, d_m) \xrightarrow{f} (N, d_N) \xrightarrow{g} (Q, d_Q).$$

Suppose f, g are continuous. Consider $(p_n) \in M$ such that $\lim_{n\to\infty} p_n = p \in M$. Then by sequential continuity $\lim_{n\to\infty} f(p_n) = f(p)$ and $\lim_{n\to\infty} g(f(p_n)) = g(f(p))$, Therefore $(p_n) \to p \implies (g \circ f)(p_n) \implies (g \circ f)(p)$. Hence $g \circ f$ preserves sequential convergence is therefore convergent. $\square$

4 Mon 9/21

Recall that we are left with continuous $\iff$ sequentially continuous.

Theorem 11: Continuity and Sequential Continuity

$f : M \to N$ is continuous if and only if it is sequentially continuous.

Proof

First recall the definitions:

(1) **Continuous** if for each $p \in M$, and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{for any } x \in M \text{ such that } d_M(x, p) < \delta, \text{ we have } d_N(fx, fp) < \epsilon.$$

(2) **Sequentially continuous** if f takes convergent sequences in M to convergent sequences in N, with limits mapping to limits.

Back to the proof:

(1) For $\implies$, suppose f is continuous and we will show it is sequentially continuous. Let (p_n) be a sequence in M such that $(p_n) \to p \in M$. We want to show that given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geqslant N \implies d_N(f(p_n), fp) < \epsilon$.

By continuity of f, we know that there exists $\epsilon > 0$ such that $d_M(x, p) < \delta$ implies $d_N(fx, fp) < \epsilon$.

On the other hand, by the convergence of (p_n), there exists $N \in \mathbb{N}$ such that $d_M(p_n, p) < \epsilon$ whenever $n \geqslant N$.

Combining these two, we can set x as such p_n with $n \geqslant N$. Then we have satisfied the ϵ condition, so $f(p_n) \to fp$.

(2) Now for $\impliedby$, assume f is sequentially continuous. Suppose that f is not continuous at p. Then there exists $\epsilon > 0$ such that $d_N(fx, fp) \geqslant \epsilon$ regardless of $d_M(x, p)$: for all $\delta > 0$ we always have $d_M(x, p) < \delta$ but $d_N(fx, fp) \geqslant \epsilon$.

Then we can construct a sequence (p_n) such that $d_M(p_i, p) < \delta = 1/i$. Clearly (p_n) converges to p, but $f(p_n)$ does not converge to fp. Contradiction.

Therefore f is continuous if and only if it is sequentially continuous. □

Remark

Soon we will introduce another equivalent condition: open sets condition.

Open and Closed Sets (in a Metric Space)

Definition 12

Let (M, d_M) be a metric space. Define $S \subset M$ be a subset. We say $p \in M$ is a **limit** of $S \subset M$ if there is a sequence $(p_n) \in S$ that converges to p.

Example 4.1

Let $M = \mathbb{R}$ and d_M be the standard metric. Let $S = \{1/n \mid n \in \mathbb{N}\}$. Then 0 is the limit of $(1, 1/2, 1/3, \ldots)$, whereas any element of the set is also a limit of the "boring" sequence containing only itself, e.g., $1/2$ is the limit of the sequence $(1/2, 1/2, 1/2, \ldots)$. Simply put, the set of all limits in this example is $S \cup \{0\}$.

Definition 13

Let (M, d_M) be a metric space. A $S \subset M$ is **closed** if it contains all its limits.

Example 4.2

Any singleton in $\mathbb{R}$ is closed. Also $\mathbb{R}$ and $\varnothing$. More general subsets of $\mathbb{R}$ include intervals like $[0, 1]$ and $[0, \infty)$. Non-examples include $[0, 1)$ since 1 is a limit.

If we let $M = \mathbb{Z}$ then *all* subsets of M are closed, since the "boring" sequences are the only kind that converges in $\mathbb{Z}$.

Definition 14

Let (M, d_M) be a metric space. A subset $S \subset M$ is **open** if for any point $p \in S$, there exists $\epsilon > 0$ such that

$$d_M(p, s) < \epsilon \implies x \in S.$$

Example 4.3

The (closed) interval $[0, 1]$ is not an open subset of $\mathbb{R}$ because $1 + \epsilon \notin [0, 1]$ for all $\epsilon > 0$. On the other hand,

$(0, 1)$ is an open subset of $\mathbb{R}$.

Theorem 15

Openness is dual to closedness: the complement of an open subset is closed, and vice versa, i.e., if $S \subset M$ is open, then $M \setminus S$ is closed, and vice versa.

Remark

A subset of a metric space can be neither open nor closed: $M = \mathbb{R}$ and $S = [0, 1)$ for example. It is not closed because 1 is not contained, and it is not open because $0 - \epsilon \notin [0, 1)$ for all $\epsilon > 0$.

Also consider $\mathbb{Q}$ in $\mathbb{R}$. It is not closed because it does not contain all its limits: consider the decimal expansion of π. It is not open because, given a rational, there always exist arbitrarily close irrationals in its neighborhood. Hence $\mathbb{Q}$ in $\mathbb{R}$ is neither closed nor open.

On the other hand, $\varnothing$ is closed as said above, so $\mathbb{R} \setminus \varnothing$ should be open. The same logic says $\mathbb{R}$ is closed and $\varnothing$ is open. Therefore they are both open and closed, also known as **clopen**.

5 Tue 9/22 Discussion

Recall:

(1) Definition of a **metric space** (M, d).

(2) To be a metric (distance function), d needs to be positive definite, symmetric, and it also needs to satisfy triangle inequality.

(3) Example: $(\mathbb{R}^n, d_{\text{st}})$: the n-dimensional Euclidean space with the standard Euclidean metric. This defines a metric space. Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$. The distance is defined by

$$d = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

(4) Discrete metric space: (M, d) with d defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

sometimes useful for constructing examples or counterexamples.

(5) Let (M, d) be a metric space. (p_n) is called a **sequence** if $p_n \in M$ for all $n \in \mathbb{N}$.

(6) A sequence is **Cauchy** in M if: given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m, n \geqslant N$ then $d_M(p_m, p_n) < \epsilon$.

(7) A function $f : M \to N$ is called **continuous** if for any $(p_n) \to p \in M$, we have $f(p_n) \to fp \in N$.

(8) Example: let (M, d) be the discrete metric space, and (N, d) be any metric space. Show that $f : M \to N$ is continuous.

Proof. Take any $(p_n) \to p$. By the definition of convergence, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geqslant N \implies d(p_n, p) < \epsilon$. If we let $\epsilon = 1/2$, then the only p_n's possible are those satisfying $p_n = p$ because we are in a discrete metric space. Therefore if (p_n) converges to p in M then its tail will simply be a bunch of p. This implies that $(f(p_n))$ has a tail consisting of only fp. Hence $(f(p_n)) \to fp$. □

Definition 16

Let (M, d_M) and (N, d_N) be two metric spaces. A function $f : M \to N$ is called a **homeomorphism** if it is bijective and bicontinuous, i.e., it satisfies the following properties:

(1) f is a bijection,

(2) f is continuous, and

(3) f^{-1} [it exists because f is bijective] is also continuous.

If such f exists, we say M and N are **homeomorphic**, and we denote this as $M \cong N$.

More reviews:

(9) Let (M, d) be a metric space. We say $S \subset M$ is **open** if for all $p \in S$, there exists $r > 0$ such that

$$B_r(p) \subset S \text{ (ball/disk with radius } r)$$

or, alternatively, $x \in M_r p \implies x \in S$.

(10) Let (M, d) be a metric space. We say $S \subset M$ is **closed** if S contains all its limit points. Recall that p is called a limit point of S if there exists a sequence $(p_n) \in S$ with $(p_n) \to p$.

(11) Let (M, d) be a metric space. $S \subset M$ is open if and only if S^c is closed.

Example 5.1

Let $(\mathbb{R}, d)$ be the standard Euclidean metric space, and let $S \subset \mathbb{R}$ be a nonempty subset bounded from above. Recall that S has a supremum such that for all $\epsilon > 0$, there exists $a \in S$ such that

$$\sup(S) - \epsilon \leqslant a \leqslant \sup(S).$$

Show that $\sup(S)$ is a limit point of S.

Solution

Let $\epsilon = \frac{1}{n}$ with $n = 1, 2, \ldots$. Then by the property listed above, there always exists $a_n \in S$ satisfying

$$\sup(S) - \frac{1}{n} \leqslant a_n \leqslant \sup(S) \implies |a_n - \sup(S)| \leqslant \frac{1}{n}$$

Hence $(a_n) \to \sup(S)$ and therefore $\sup(S)$ is a limit point of S.

Worksheet #5

Problem 7

Let (M, d) be a metric space. Show that for any $r > 0$, and for any $p \in M$, the open ball $B_r(p)$ (or $M_r p$) is an open subset of M.

Solution

To show the ball is open, we need to verify by definition. Take any $x \in M_r p$. Then $d(x, p) < r$. Take $r' = r - d(x, p)$. Then, $M_{r'} x \subset M_r p$ because for any point $x' \in M_{r'} x$ we have $d(x', x) < r'$. Then by triangle inequality

$$d(p, x') \leqslant d(p, x) + d(p, p') < r.$$

Problem 8

A space M is equipped with the discrete metric d.

(1) Verify that (M, d) is a metric space.

(2) Show that any subset $E \subset M$ is both open and closed, “clopen”.

Solution

The first part is omitted since it's part of HW5.

For the second part, if we want to show the subset is open, we need to show that for all $p \in E$, there exists $r > 0$ such that $x \in M_r p \implies x \in E$. Letting $r = 1/2$ will do the trick since, for a discrete metric space, $M_{(1/2)} p = \{p\}$. Likewise, since E is arbitrary, we can apply the same logic and claim that E^c is open. Then $E = (E^c)^c$ is also closed.

Problem 9

Let (M, d) be a metric space.

(1) Let p, q, s, t be any four points in M. Show that

$$d(p,t) \leqslant d(p,q) + d(q,s) + d(s,t).$$

(2) Suppose (x_n) and (y_n) are two Cauchy sequences in M, and let $r_n = d(x_n, y_n)$ be the sequence in $\mathbb{R}$ consisting of distances between their respective terms. Show that (r_n) converges in $\mathbb{R}$.

Solution

(1) The first inequality is immediate if we use the triangle inequality twice:

$$d(p,t) \leqslant d(p,s) + d(s,t) \leqslant d(p,q) + d(q,s) + d(s,t).$$

(2) Suppose $(x_n) \to x$ and $(y_n) \to y$. For $m, n \in \mathbb{N}$, from part (1) we have

$$r_n = d(x_n, y_n) \leqslant d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) = d(x_n, x_m) + r_m + d(y_n, y_m).$$

Then the Cauchy of both sequences implies we can set an arbitrary $\epsilon/2$ and find m, n satisfying

$$d(x_n, x_m) < \frac{\epsilon}{2} \text{ and } d(y_n, y_m) < \frac{\epsilon}{2}.$$

Then

$$|r_n - r_m| < d(x_n, x_m) + d(y_n, y_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

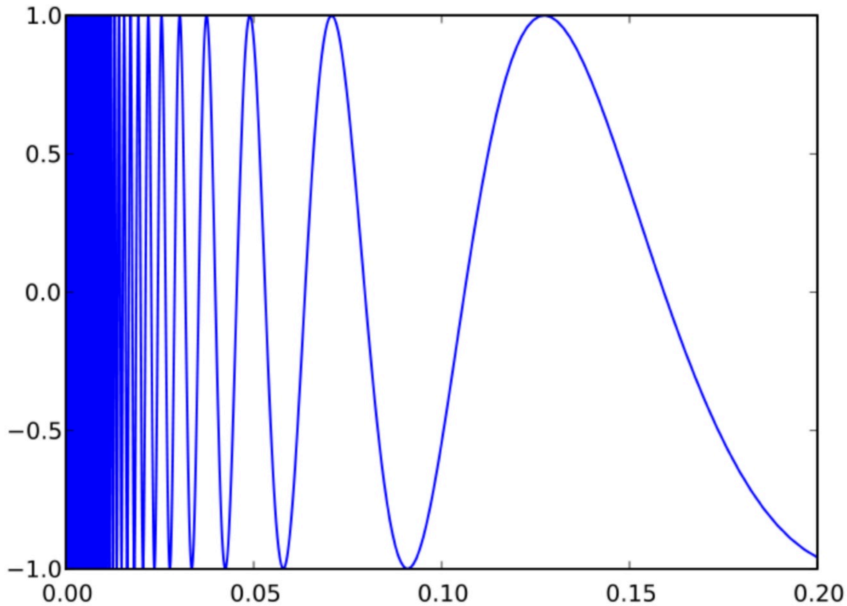
Hence (r_n) is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, we conclude (r_n) is convergent in $\mathbb{R}$.

6 Wed 9/23

Recall from last class: definition of open and closed subsets: open if the subset contains all limits, and closed if the subset satisfies that $M_r p/B_r(p)$ condition — that the open subset has no boundary and we can always fit a tiny "ball" around a point in the subset no matter how close it is to the "boundary".

Example 6.1: Topologist's Sine Curve

Consider $M = \mathbb{R}^2$ and $S = \{(x, \sin(1/n) \mid x \in (0,1]\} \cup \{(0,0)\}$. The graph is a sinusoidal wave but with ever increasing frequency as $x \to 0$.



Topologist's Sine Curve, from class website

The set is not open because $(x, \sin(1/n) + \epsilon) \notin S$ no matter how small ϵ is. The graph of this set is a line and it has lower dimension than $\mathbb{R}^2$.

Also consider all the points of interceptions between the graph of S and $y = 1$. The points converge to $(0,1)$ but clearly $(0,1) \notin S$.

To make this set closed, consider revising it as

$$S = \{(x, \sin(\frac{1}{x})) \mid x \in (0,1]\} \cup \{(0,y\} \mid y \in [-1,1].$$

Definition 17

Given any subset $S \subset M$, its **closure**, denoted by $\overline{S}$, is given by the set of all limits of S.

Remark

This set is closed. In particular, this is the *smallest* closed subset of M containing S.

Example 6.2

$\overline{\mathbb{Q}} = \mathbb{R}$.

Now we will give a proof for a theorem mentioned in last class:

Theorem 18

Openness is dual to closedness.

Proof

Suppose $S \subset M$ is an open subset. We want to show that $S^c := M \setminus S$ is closed. Let (p_n) be a sequence in S^c. It has to converge to some $p \in M$. We want to show that $p \in S^c$. Suppose not, then $p \in S$, and by the openness of S and the convergence of (p_n) we have

$$\text{there exists } \epsilon > 0 \text{ and } m \in \mathbb{N} \text{ such that } \begin{cases} d(p_m, p) < \epsilon \text{ (convergence of } (p_n)) \\ p_m \in M_\epsilon(p) \implies p_m \in S \text{ (openness)} \end{cases}$$

and this contradicts our assumption that $(p_n) \in S^c$. Hence S is closed.

Now suppose $S \subset M$ is closed. We want to show S^c is open. By definition we want to show that for all $p \in S^c$ there exists $\epsilon > 0$ such that $M_\epsilon p \subset S^c$. Suppose not, then for some point $p \in S^c$, for all $\epsilon > 0$, we always have $M_\epsilon p \cap S \neq \varnothing$ since we cannot find any balls enclosing p that's completely in S^c. Then if we set $\epsilon = 1/n$ with $n = 1, 2, \dots$, we can find a sequence of points from $M_\epsilon p \cap S$ as ϵ changes. Then this sequence converges to p since $(1, 1/2, 1/3, \dots)$ converges to 0. However, $p \in S^c$, so S does not contain all limit points. Contradiction. Therefore S^c must be open. □

7 Fri 9/25

Theorem 19

Let (M, d) be a metric space. The set of all open subsets of M satisfies the following:

(1) any arbitrary union of open subsets is again open.

(2) any intersection of finitely many open subsets is open.

(3) M and $\varnothing$ are both open.

Remark

A **topological space** is a set M along with a collection of subsets satisfying the properties above. We are not talking about metric here — we are talking about the collection of subsets, the open subsets of M.

By above, given any metric space (M, d), M, along with the collection of all open subsets of M, is a topological space.

The collection of open subsets of M is called the **topology on** M.

A quick digression:

Definition 20: Topological Continuity

A function $f : M \to N$ is *topologically continuous* if, given $U \in N$ is open, then $f^{-1}(U) \subset M$ is open.

Theorem 21: Topological Continuity (Open Set)

A function f is continuous if it is topologically continuous. Proven in HW6.

Back to Theorem 19.

Proof of Theorem 19

We've already seen that M and $\varnothing$ are open.

Now, let $S_{\alpha \in I}$ be a collection of open subsets of M. (Here I is some indexing set, not necessarily countable.)

We want to show that

$$S := \bigcup_{\alpha \in I} S_\alpha \text{ is an open subset of } M.$$

Consider a point $p \in S$. We want to show that there exists $\epsilon > 0$ such that $M_\epsilon p \subset S$. Since $p \in S$, it must be true that $p \in S_{\alpha_0}$ for some $\alpha_0 \in I$. Since S_{α_0} is open, there exists $\epsilon_0 > 0$ such that $M_{\epsilon_0} p \subset S_{\alpha_0}$. Therefore $M_{\epsilon_0} p \subset S_{\alpha_0} \subset S$, which completes the proof of part (1).

For part (2), suppose $S_1, S_2, \ldots, S_n \in M$ are open. Define

$$S := \bigcap_{i \in [1,n]} S_i.$$

Pick a $p \in S$. Then $p \in S_i$ for all $i \in [0, n]$. Since each S_i is open, there exists ϵ_i such that $M_{\epsilon_i} p \subset S_i$. Then, choosing ϵ', the minimum of all these ϵ_i's, it is guaranteed that $M_{\epsilon'} p \subset S_i$ for all $i \in [1, n]$. Hence $M_{\epsilon'} p \subset S$, and this completes the proof that S is set. □

Remark

For part (2), if we have infinitely many sets, then the ϵ_i's may lead to $\inf_i (\epsilon_i) = 0$, in which case the proof would break down.

Example 7.1

Consider

$$\bigcap_{n=1}^{\infty} (-\frac{1}{n}, 1 + \frac{1}{n}) = [0, 1], \text{ closed!}$$

Corollary 22

An arbitrary intersection of closed subsets is closed, and a union of finitely many closed subsets is closed.

The proof can be easily constructed using the theorem above, the duality of openness and closedness, and De Morgan's law.

For the counterexample analogy, consider

$$\bigcup_{r \in (0,1)} \{r\} = (0, 1), \text{ infinite union of closed subsets become open..}$$

Definition 23

A map f from (M, d_M) to (N, d_N) is an **isometry** if

(1) f is a bijection, and

(2) for $p, q \in M$, we always have $d_M(p,q) = d_N(fp, fq)$.

Definition 24

A map f from $(M, d_M), (N, d_N)$ is a **homeomorphism** if f is bijective and bicontinuous. (See the previous discussion notes).

Remark

Being homeomorphic is a weaker statement as compared to being isometric. Being isometric implies being homeomorphic but not the converse.A homeomorphism sets up a one-to-one correspondence between open subsets in M and those in M.

Example 7.2

(From HW5) $[0,1] \cong [0,2]$ but they are not isometric. $f : x \mapsto 2x$ is bijective and bicontinuous but there does not exist a distance preserving function between these two sets.

Example 7.3

A, $\text{\Large A}$, and $\mathcal{A}$ are pairwise homeomorphic but not isometric. A and $\mathfrak{A}$ are not neither homeomorphic nor isometric. A and ᐊ are both homeomorphic and isometric.

8 Mon 9/28

To generalize Cauchy sequences from $\mathbb{R}$ to a metric space:

Definition 25

A sequence (p_n) in a metric space (M, d) is **Cauchy** if, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d_(p_m, p_n) < \epsilon \text{ for all } m, n \geqslant N.$$

Definition 26

A metric space is **complete** if every Cauchy sequence converges.

Example 8.1

$\mathbb{R}^n$ is complete. Think of each point in $\mathbb{R}^n$ as an n-tuple. Then if (p_n) is Cauchy, each component of (p_n) is Cauchy, and we know $\mathbb{R}$ is complete. Hence each component converges and (p_n) converges.

Remark

The Cartesian product of complete metric spaces is complete.

Example 8.2

$(0, 1)$, with the induced metric from $\mathbb{R}$, is not complete. The sequence (p_n) with $p_i = 1/2^i$ is Cauchy but it doesn't converge to anything in this metric space.

Example 8.3

$\mathbb{Z}$ (with the induced metric from $\mathbb{R}$) is complete. Setting $\epsilon = 1/2$ we see that each Cauchy sequence eventually has a constant tail, and that constant is in $\mathbb{Z}$. Hence $\mathbb{Z}$ is complete.

On the other hand, $\mathbb{Q}$ is not complete. Consider, again, the π decimal expansion example.

Remark

If $\mathbb{Z}$ and $\mathbb{Q}$ were *equipped with discrete metric*, they become complete! Therefore metric matters.

Definition 27

A metric space (M, d) is **connected** if the only clopen subsets are M and $\varnothing$. In other words, it does *not* have any proper clopen subset.

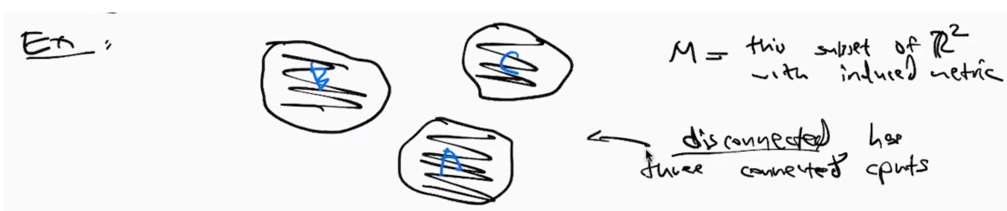
A metric space is **disconnected** if it has a proper clopen subset.

Example 8.4

$(0,1) \cup (2,3)$ is disconnected. For example, $(0,1)$ is clopen. It is open for fairly obvious reasons. It's tempting to think that a sequence in $\mathbb{R}$ converging to 1 is not convergent in this case, but if we treat $M = (0,1) \cup (2,3)$, then $1 \notin M$, which means such sequence doesn't even converge in the metric space. In fact since $(2,3)$ is open, its complement $(0,1)$ must be closed.

Example 8.5

Each "blob" is a clopen subset of M.



From Siegel's Lecture

Example 8.6

What about $\mathbb{Q}$ with the induced metric from Euclidean space?

Consider $U \subset \mathbb{Q}$ given by

$$U = \{r \in \mathbb{Q} : r < 0 \text{ or } r^2 < 2\}.$$

It follows that U is open for obvious reasons, and U is closed because $\sqrt{2} \notin \mathbb{Q}$. Hence it's open, and $\mathbb{Q}$ is disconnected.

Definition 28

We say something is a "**topological property**" of a metric space if it is preserved under homeomorphisms.

Problem 10

Is completeness a topological property of a metric space? If (M, d_M) and (N, d_N) are homeomorphic and (M, d_M) is complete, is (N, d_N) necessarily complete?

Solution

No. $\mathbb{R}$ is complete and $(0, 1)$ is not complete, but the *Signoid function*

$$G(x) = \frac{e^x}{1 + e^x}$$

is a bijective bicontinuous function. Hence it's a homeomorphism, and completeness is *not* a topological property.

Problem 11

Is connectedness a topological property?

Solution

Yes! A homeomorphism $f : M \to N$ would map proper clopen subsets of M to proper clopen subsets of N.

Theorem 29

Every closed subset of a complete metric space is itself a complete metric space (with the induced metric).

Theorem 30: Inheritence Principle

Let (M, d) be a metric space, and $N \subset M$ a subspace with the induced metric. A subset $U \subset N$ is open (in N) if and only if $U = N \cap V$ for some open subset $V \subset M$. Likewise for the condition to be closed.

Example 8.7

Put $M = \mathbb{R}^2$ and $N = \{(x, \sin(1/x) \mid x \in (0,1)\}$ the topologist's sine curve. If we put

$$V = \{(x, \sin(\frac{1}{x}) \mid x \in (0,1) \text{ such that } x^2 + \sin^2(\frac{1}{x}) \leqslant \frac{1}{2}\}$$

is it a closed subset of N?

Solution

Yes! If we define

$$V := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant \frac{1}{2}\}$$

then it represents a disk with radius $1/\sqrt{2}$. Then $U = N \cap V$.

Clearly the disk V is an closed subset of $\mathbb{R}^2$ (why?). By the inheritance principle, U is a closed in N.

9 Tue 9/29 Discussion

Definition 31

A topology on M is the collection $\mathcal{T}$ of subsets of M satisfying

(1) $\varnothing, M \in \mathcal{T}$,

(2) arbitrary union of sets in $\mathcal{T}$ is in $\mathcal{T}$, and

(3) finite intersection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

Example 9.1

Let $M = \{1, 2, 3\}$ and define

$$\mathcal{T}(M) = \{\varnothing, \{1\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Theorem 32

Given (M, d), a metric space, the collection of open sets forms a topology on M. (This is immediate from what we've done last class.)

Remark

Given a metric space, we get a topology *induced by the metric*. Such topology is called **metrizable**.

Example 9.2

The example above with $M = \{, 1, 2, 3\}$ and

$$\mathcal{T}(M) = \{\varnothing, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$$

is not a metrizable topology. It is not induced by any metric because any singleton in a metric is closed, so $\{3\}$ is closed and $\{1, 2\}$ is open which is not in $\mathcal{T}(M)$.

Corollary 33

Given (M, d), the following are equivalent:

(1) arbitrary union of open sets is open in (M, d),

(2) finite intersection of open sets is open in (M, d),

(3) arbitrary intersection of closed sets is closed in (M, d), and

(4) finite union of closed sets is closed in (M, d).

Definition 34

Let $(M, \mathcal{T}_M)$ and $(N, \mathcal{T}_N)$ be topological spaces (set with topology), we say $f : M \to N$ is **topologically continuous** if the $E \in \mathcal{T}_N$ is open then $f^{-1}(E) \in \mathcal{T}_M$ is also open.

Theorem 35

The following are equivalent for continuity of $f : M \to N$:

(1) f preserves sequential continuity.

(2) f satisfies the $\epsilon - \delta$ condition.

(3) f is topologically continuous:

  (1) the preimage of an open set is open, or

  (2) the preimage of a closed set is closed.

A quick recap of the inheritance principle:

Theorem 36: Inheritence Principle

Let (M, d) be a metric space, and $N \subset M$ a subspace with the induced metric. A subset $U \subset N$ is open (in N) if and only if $U = N \cap V$ for some open subset $V \subset M$. Likewise for the condition to be closed.

Example 9.3

Let $M = \mathbb{R}$ and d the standard Euclidean metric. Let $N = [0, 1)$. Is it open in M? In N?

Solution

Not in M because no ball containing 0 is also a subset of this interval. However, it is open in N because

$$[0, \frac{1}{2}) = \underbrace{(-1, \frac{1}{2})}_{\text{open subset of } \mathbb{Q}} \cap \underbrace{[0, 1)}_{N \text{ itself, open}}$$

Quick recap of connectedness:

Definition 37

Let (M,d) be a metric space. M is called **disconnected** if M can be written as $M = A \sqcup A^c$ where A and A^c are proper clopen sets.

Example 9.4

$M = (-1,1) \cup (2,3)$ with standard metric is disconnected.

If we let $A = (-1,1)$, then A is open in M and A^c is closed. Likewise, A^c is open and A is closed. Hence A, A^c are proper clopen sets in M.

Example 9.5

$\mathbb{Q}$ is disconnected because it can be written as

$$\mathbb{Q} = \{r \in \mathbb{Q} \mid r < \sqrt{2}\} \cup \{r \in \mathbb{Q} \mid r > \sqrt{2}\}.$$

Definition 38

(M,d) is called **connected** if it is not disconnected.

Theorem 39

If $f : M \to N$ is continuous and surjective, then if M is connected we have N is connected See Pugh p.87.

10 Tue 9/29 Discussion

Definition 40

A topology on M is the collection $\mathcal{T}$ of subsets of M satisfying

(1) $\varnothing, M \in \mathcal{T}$,

(2) arbitrary union of sets in $\mathcal{T}$ is in $\mathcal{T}$, and

(3) finite intersection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

Example 10.1

Let $M = \{1, 2, 3\}$ and define

$$\mathcal{T}(M) = \{\varnothing, \{1\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Theorem 41

Given (M, d), a metric space, the collection of open sets forms a topology on M. (This is immediate from what we've done last class.)

Remark

Given a metric space, we get a topology *induced by the metric*. Such topology is called **metrizable**.

Example 10.2

The example above with $M = \{, 1, 2, 3\}$ and

$$\mathcal{T}(M) = \{\varnothing, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$$

is not a metrizable topology. It is not induced by any metric because any singleton in a metric is closed, so $\{3\}$ is closed and $\{1, 2\}$ is open which is not in $\mathcal{T}(M)$.

Corollary 42

Given (M, d), the following are equivalent:

(1) arbitrary union of open sets is open in (M, d),

(2) finite intersection of open sets is open in (M, d),

(3) arbitrary intersection of closed sets is closed in (M, d), and

(4) finite union of closed sets is closed in (M, d).

Definition 43

Let $(M, \mathcal{T}_M)$ and $(N, \mathcal{T}_N)$ be topological spaces (set with topology), we say $f : M \to N$ is **topologically continuous** if the $E \in \mathcal{T}_N$ is open then $f^{-1}(E) \in \mathcal{T}_M$ is also open.

Theorem 44

The following are equivalent for continuity of $f : M \to N$:

(1) f preserves sequential continuity.

(2) f satisfies the $\epsilon - \delta$ condition.

(3) f is topologically continuous:

  (1) the preimage of an open set is open, or

  (2) the preimage of a closed set is closed.

A quick recap of the inheritance principle:

Theorem 45: Inheritence Principle

Let (M, d) be a metric space, and $N \subset M$ a subspace with the induced metric. A subset $U \subset N$ is open (in N) if and only if $U = N \cap V$ for some open subset $V \subset M$. Likewise for the condition to be closed.

Example 10.3

Let $M = \mathbb{R}$ and d the standard Euclidean metric. Let $N = [0, 1)$. Is it open in M? In N?

Solution

Not in M because no ball containing 0 is also a subset of this interval. However, it is open in N because

$$[0, \frac{1}{2}) = \underbrace{(-1, \frac{1}{2})}_{\text{open subset of } \mathbb{Q}} \cap \underbrace{[0,1)}_{N \text{ itself, open}}$$

Quick recap of connectedness:

Definition 46

Let (M, d) be a metric space. M is called **disconnected** if M can be written as $M = A \sqcup A^c$ where A and A^c are proper clopen sets.

Example 10.4

$M = (-1, 1) \cup (2, 3)$ with standard metric is disconnected.

If we let $A = (-1, 1)$, then A is open in M and A^c is closed. Likewise, A^c is open and A is closed. Hence A, A^c are proper clopen sets in M.

Example 10.5

$\mathbb{Q}$ is disconnected because it can be written as

$$\mathbb{Q} = \{r \in \mathbb{Q} \mid r < \sqrt{2}\} \cup \{r \in \mathbb{Q} \mid r > \sqrt{2}\}.$$

Definition 47

(M, d) is called **connected** if it is not disconnected.

Theorem 48

If $f : M \to N$ is continuous and surjective, then if M is connected we have N is connected See Pugh p.87.

Worksheet # 7

Problem 12

Give a metric space $(M.d)$ and $S \subset M$, define

$$\lim(S) = \{p \in M \mid p \text{ is a limit point of } S\}.$$

Show that

$$\lim(S) = \bigcap_{V \supset S, V \text{ closed}} V.$$

Note that the RHS is called the closure of S in topology.

Solution

We will show mutual inclusion. First, LHS $\subset$ RHS. Pick $p \in \lim(S)$. By definition there exists a sequence $(p_n) \to p$. Since $S \subset V$, we know (p_n) is also in V. Since each V is closed, we know $p \in \bigcup V$.

Now to show RHS $\subset$ LHS: first notice that $\lim(S)$ is closed (see Pugh p.68), and we know $S \subset \lim(S)$ since for any element $s \in S$, the sequence $(s, s, \dots)$ converges to $s \in \lim(S)$. Now notice that $\lim(S)$ meets the requirement to be a V, and let's call it V_i. Hence $\bigcap V \subset V_i = \lim(S)$.

Problem 13

Prove or disprove the following:

(1) If A and B are connected, then so is $A \cap B$.

(2) If A and B are connected, then so is $A \cup B$.

Solution

(1) No. Consider $\{(x,y) \mid y = 0\} \cup \{(x,y) \mid x^2 + y^2 = 1\}$.

(2) Even more obvious. No.

Remark

A metric space (M,d) with finitely many elements is disconnected.

Problem 14

Definition: given metric space (M,d), a subset $E \subset M$ is called **dense** if for all $x \in M$, either $x \in E$ or x is a limit point of E.

Let f be a continuous function from one metric space (M,d_M) to another metric space (N,d_N), and let E be a dense subset of M. Prove that $f(E)$ is dense in $f(M)$.

Remark

To visualize "dense", think of $\mathbb{Q}$ in $\mathbb{R}$. Pick $x \in \mathbb{R}$. It is either rational or irrational, namely $x \in \mathbb{Q}$ or $x \in \mathbb{R} \setminus \mathbb{Q}$. If it is irrational, we can always find a rational within $(x, x+1/n)$. Then x becomes a limit point.

Solution

We want tos how that for any $y \in f(M)$, either $y \in f(E)$ or y is a limit poinf of $f(E)$.

Now for such $y \in f(M)$, there exists $x \in M$ such that $f(x) = y$. Since E is dense in M, either $x \in E$ or x is a limit point of E.

(1) If $x \in E$ then $y \in f(E)$ and we are done.

(2) If x is a limit point of E then there exists $(x_n) \to x$. By the sequential continuity of f we know $f(x_n) \to f(x) = y$. Since each $f(x_i) \in f(E)$ we know y is the limit of a sequence in $f(E)$ and therefore is a limit point of $f(E)$.

11 Wed 9/30

Definition 49: Compactness

A metric space (M,d) is **sequentially compact** if every subsequence in M has a convergent subsequence.

Remark

Later, we will give an equivalent definition for compactness: "covering compact". For now we will focus on sequential compactness.

Definition 50

A metric space (M, d) is **bounded** if there exists $R > 0$ such that $d_M(x, y) < R$ for all $x, y \in M$. In other words, the distance between any two points is bounded.

Lemma 11.1

A compact metric space is bounded.

Proof of lemma

Assume by contradiction that (M, d) is unbounded. Then we can find $y \in M$ and a sequence (x_n) such that $d_M(y, x_i) \geqslant i$. Then the sequence $(y, x_1, x_2, \dots)$ does not have any convergent subsequence. □

Example 11.1

$\mathbb{R}^n$ is not compact.

Example 11.2

If (M, d) is a finite metric space then it is compact because for each sequence we can find a constant sequence.

Definition 51

If (M, d) is a metric space, we say that a subset $S \subset M$ is compact if S with the induced metric is compact.

Lemma 11.2

Any compact subset of a metric space is closed.

Example 11.3

The open interval $(0, 1)$ is not compact. Any sequence converging to 1 fails to converge in this interval, and so are any subsequences.

Proof of Lemma

Assume $S \subset M$ is compact and let (p_n) be a sequence such that $(p_n) \to p \in M$. Since S is compact, we know that there exists a subsequence (p_{n_i}) converges to $s \in S$. But we also know the subsequence must converge to the same limit as does the mother sequence, we know $p = s$ and $p \in M$. Hence S is closed. □

Problem 15

Does closed and bounded necessarily imply compact?

Theorem 52: Heine-Borel Theorem

Any closed and bounded subset of $\mathbb{R}^m$ is compact.

Solution

In general, closed and boundedness do not imply compactness. Consider $\mathbb{N}$ equipped with the discrete metric. Since the distance in a discrete metric is at most 1, every subset is bounded. Let $S \subset \mathbb{N}$ be the closed subset. The sequence $(1, 2, \ldots)$ cannot possibly have a convergent subsequence so $\mathbb{R}$ equipped with discrete metric cannot be compact.

Theorem 53

$[a, b]$ (with standard metric) is compact.

Proof

Pick an (infinite) sequence (x_n). First divide $[a, b]$ into two equal halves with $L_1 = [a, a + (b - a)/2]$ and $R_1 = [a + (b - a)/2, b]$. Since (x_n) is infinite, at least one of these intervals contain infinitely many terms of this sequence. Choose an interval that contains infinitely many terms, call it I_1, and proceed with this bisection to get $I_2, I_3, \ldots$. Now we can construct a (sub)sequence (x_{n_k}) such that $x_{n_i} \in I_i$. Then for $i, j \geqslant N$, we have

$$|x_{n_i} - x_{n_j}| \leqslant \frac{b-a}{2^N}$$

which shows that the subsequence (x_{n_k}) is Cauchy. Since $\mathbb{R}$ is complete, it converges in $\mathbb{R}$. Therefore $[a, b]$ is compact. □

Remark

Pugh gives an alternate proof in his book. Pg. 79.

Theorem 54

The Cartesian product of two compact sets is compact. (This can be generalized by induction.)

Proof

Let $A \subset M$ and $B \subset N$ be given. Pick $(a_n) \in A$ and $(b_n) \in B$. The compactness of A implies (a_n) has a convergent subsequence (a_{n_k}) that converges to $a \in A$. Fix these indexes. It's obvious that (b_{n_k}) is also a sequence in B, so it must also have a subsequence $(b_{n_k(\ell)})$ that converges to $b \in B$. Now $(a_{n_k(\ell)})$ is a subsequence of the convergent sequence (a_{n_k}) and must also converge to a. Then

$$(a_{n_k(\ell)}, b_{n_k(\ell)}) \to (a, b)$$

and this shows $A \times B$ is compact. □

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Definition 55

A subset $S \subset M$ of a metric space (M, d) is compact if every sequence in S has a subsequence which converges to a limit in S.

Recall from last class that we've shown that

(1) compact $\implies$ closed and bounded, but

(2) closed and bounded $\not\Longrightarrow$ in general.

(3) However, Heine-Borel Thm states that closed and bounded in $\mathbb{R}^n \implies$ compact.

Alternate proof to show closed intervals are compact

Suppose we are trying to show $[a, b]$ is compact. Let (p_n) be a sequence in $[a, b]$. Define

$$C := \{x \in [a, b] \mid p_i < x \text{ for only finitely many } i\text{'s}\}$$

Observe that C is nonempty ($a \in C$) and it is clearly bounded above by b. Now we want to show (p_n) has a subsequence that converges to $m = \sup(C)$. If not, then there exists $\epsilon > 0$ such that $p_k \in (m - \epsilon, m + \epsilon)$ only finitely many times (suppose not then we can construct a sequence of ϵ's and eventually find a subsequence converging to m). Therefore $m + \epsilon$ is also in C, contradicting m being the supremum. Hence each $(p_n) \in [a, b]$ has a convergent subsequence, and this interval is compact. $\square$

Definition 56: Product metrics

Let (M, d_M) and (N, d_N) be metric spaces. We have the following metrics:

(1) $d_{\text{sum}}((m, n), (m', n')) = d_M(m, n') + d_N(n, n')$. Example: $\mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}$ with

$$d((x, y), (x', y')) = |x - x'| + |y - y'|.$$

(2) $d_E((m, n), (m', n'))\sqrt{(d_M(m, m')^2 + d_N(n, n')^2}$. Example: Euclidean distance.

(3) $d_{\max}((m, n), (m', n')) = \max\{d_M(m, m'), d_N(n, n'\}$.

Theorem 57

Let $((m_k, n_k))$ be a sequence in $M \times N$. The ollowing are equivalent:

(1) the sequence converges with respect to d_{sum};

(2) … with respect to d_E;

(3) … with respect to $d_{\max}$;

(4) (m_k) converges in M and (n_k) converges in N.

Theorem 58

All three of these metrics are "topologically equivalent", i.e., they induce the same collection of open subsets of $M \times N$.

Recall that the L_p norm on $\mathbb{R}^n$ is given by

$$\|\mathbf{r}\|_p = \left(|r_1|^p + |_2|^p + \cdots + |r_n|^p\right)^{1/p}.$$

We also put L_∞ as

$$\|\mathbf{r}\|_\infty = \max\{|r_1|, |r_2|, \ldots, |r_n|\}.$$

Remark

All these different L_p norms induce different metrics, but they are indeed topologically equivalent. One sequence with respect to L_{p_1} converges if and only if the sequence also converges with respect to L_{p_2}.

Corollary 59

The product of any finite number of compact metric spaces is compact.

Proof

See the last proof (of sub-sub-subsequences) from class on Wed. Then use induction. □

Corollary 60

Boxes $([a_1, b_1] \times [a_2, b_2] \times [a_n, b_n])$ are compact.

Theorem 61

Any closed subset of a compact metric space is compact.

Proof

Consider $S \subset M$ (from now on we will drop the cumbersome (M, d) unless it's necessary to explicitly address it) where M is compact, and S is closed. Consider $(p_n) \in S$. By compactness of M, we know a subsequence converges to $p \in M$. But since S is closed, $p \in S$. Hence S is compact. □

Proof of Heine-Borel Thorem

Since S is closed and bounded in $\mathbb{R}^n$. We know $\mathbb{R}^n$ is compact because it is the Cartesian product of n compact metric spaces. Then S is a closed subset of this compact metric space and is therefore compact. □

13 Mon 10/5

Problem 16

Consider a sequence $(p_n) \in \mathbb{R}^2$ defined by $p_k = (\cos^2(k), 3\cos(k) + \sin^3(7k)$. Does it have a convergent subsequence?

Solution

The sequence is contained in the box $[0, 1] \times [-4, 4]$. By Heine-Borel, the box is compact, so the sequence must have a convergent subsequence.

Theorem 62: Bolzano-Weierstrass Theorem

Any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Theorem 63

The image of a compact set under a continuous function is compact.

Proof

Consider $f : M \to N$, two metric spaces, with (M, d_M) compact. We want to show that $f(M)$ is sequentially compact.

Consider a sequence (p_n) in M; it suffices to show that $(f(p_n))$ has a convergent sequence. This is immediate since (p_n) has a convergent by the compactness of M and that f preserves sequential convergence. Hence the image of that subsequence, a subseqeuence of $(f(p_n))$, converges in N. $\square$

Corollary 64

Let (M, d) be a compact metric space. Then any continuous function $f : M \to \mathbb{R}$ assums its minimum and also its maximum.

Proof

The image is $f(M) \subset \mathbb{R}$. By the theorem above, M is compact, so $f(M)$ is a compact subset of $\mathbb{R}$. Recall that compact $\implies$ closed and bounded. Set $A := \inf(f(M))$ and $B := \sup(f(M))$. Since $f(M)$ is closed, it contains all its limit points, and thus $A, B \in f(M)$. $\square$

Remark

We are only saying that f continuous and M compact together imply $f(M)$ is compact. Not the other way around.

(1) Non-compact to non-compact: the identity function $f : \mathbb{R} \to \mathbb{R}$, or the Signoid function $f(x) = e^x/(1+e^x)$

(2) Non-compact to compact: the sine function $f : \mathbb{R} \to [-1, 1]$ by $x \mapsto \sin(x)$.

(3) Compact, but not continuous, to non-compact: the piecewise function $f : [-\pi/2, \pi/2] \to \mathbb{R}$:

$$f(x) = \begin{cases} 0 & x = -\frac{\pi}{2}, \frac{\pi}{2} \\ \tan^{-1}(x) & \text{otherwise} \end{cases}$$

Theorem 65

Suppose $f : M \to N$ is continuous. If M is compact, and if f is a bijection, then it's a homeomorphism.

Example 13.1: Non-example

If M is not compact this can be blatantly false: let S^1 be the unit circle in $\mathbb{R}2$. Consider $f : [0, 2\pi) \to S^1$ defined by $\theta \mapsto (\cos\theta, \sin\theta)$. Clearly this function is continuous and bijective, but the inverse is not continuous, for the inverse of any sequence that approaches $(1, 0)$ should converge to 2π, while actually $(1, 0)$ corresponds to 0.

Definition 66

A **separation** of M is a decomposition $M = A \sqcup A^c$ where both sets are proper clopen.

Example 13.2

$M = (0, 1) \cup (1, 2]$. The two sets are a separation of M.

Theorem 67

If $f : M \to N$ is continuous and M is connected, then $f(M)$ is connected.

Proof

Suppose $A \subset N$ is proper clopen, then $f^{-1}(A)$ is proper clopen in M. □

Alternately, suppose $N = A \cup A^c$ is a separation of N. Then Then $M = f^{-1}(A) \cup (f^{-1}(A))^c$ which again shows M is disconnected since this is a separation of M. Hence no separation of N exists. □

14 Tue 10/6 Discussion

Proposition 68

If $f : M \to N$ is a homeomorphism, then for all $p \in M$, $M \setminus \{p\} \cong N \setminus \{f(p)\}$.

Proof

Now we strict f on $M \setminus \{p\}$ and call this g. It's still a bijection because p and its corresponding $g(p)$ are simultaneously removed. Now we want to show that g is open. Pick $E \in N \setminus \{f(p)\}$ that is open. By inheritance principle, since E is open, $E = U \cap (N \setminus \{f(p)\}$ for some U open in N. Then

$$g^{-1}(E) = g^{-1}(U \cap (N \setminus \{f(p)\}) = f^{-1}(U \cap (N \setminus \{f(p)\}) = f^{-1}U \cap f^{-1}(N \setminus \{f(p)\})$$

and by inheritance principle, $g^{-1}(E)$ is again open. Hence g is continuous. Likewise we can show g^{-1} is continuous, which then makes g a homeomorphism. □

Remark

If $M \cong N$, then for finite set $P = \{p_1, p_2, \ldots, p_n\}$, we have

$$M \setminus P \cong N \setminus f(P).$$

Theorem 69

Compactness and connectedness are preserved under homeomorphism, whereas completeness and boundedness are not necessarily preserved under homeomorphism.

Example 14.1

Consider the arctan function betweem $(0,1)$ and $\mathbb{R}$. Clearly $\mathbb{R}$ is not complete but $\mathbb{R}$ is not bounded, yet the arctan function is a homeomorphism.

Definition 70

Let (M,d) be a metric space. $K \subset M$ is called **compact** if for each $(p_n) \in K$, there exists a subsequence converging to a point $p \in K$.

Remark

Being closed and bounded do not necessarily imply being compact. Consider $\mathbb{Z}$ with discrete metric. $\mathbb{N}$ is closed and bounded since the distance between any two elements is at most 1. It is not compact because $(1,2,\dots)$ does not have a convergent subsequence. However, closed and bounded imply compact in $\mathbb{R}^n$ by Heine-Borel theorem.

Worksheet 8

Problem 17

Show that any two of $(0,1), (0,1]$, and $[0,1]$ are not homeomorphic.

Solution

To show $(0,1) \ncong [0,1]$, if we remove $\{1\} \in [0,1]$ then we have to remove some $\{x\}$ for some $x \in (0,1)$. Then the first set becomes disconnected while the second is still connected.

Likewise for $(0,1) \ncong (0,1]$ if we take away $\{1\}$ from the second interval.

For $(0,1] \ncong [0,1]$, taking off two points will always finish the proof.

Problem 18

Show that arbitrary intersection of compact sets is compact, and the union of finitely many compact sets is compact.

Solution

Suppose $(K_1, K_2, \dots)$ are compact sets. The compactness implies they are all bounded and closed. Therefore their intersection is also closed. Clearly this intersection is also a subset of compact set. Therefore it's compact.

Pick any sequence in (p_n) in the union. There exists at least one K_i from which comes infinitely many terms that constitute (p_n). Then it's a one line proof by the compactness of K_i.

Problem 19

Determine whether the following sets are compact or not.

(1) $X = \{(x, y) \in \mathbb{R}^2 \mid x \geqslant 0, y = \sin x\}$. *This is unbounded and therefore not compact by Heine-Borel.*

(2) $Y = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, the unit sphere in $\mathbb{R}^3$. *This is compact because it is closed and bounded (by Heine-Borel).*

(3) $Z = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\} \subset \mathbb{R}$. *This is compact by Heine-Borel again.*

Problem 20

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and consider $A \subset R$ and its graph $G = \{(x, f(x)) \mid x \in A\}$. Show that

(1) If A is closed, then G is closed.

(2) If A is compact, then G is compact.

Remark

If A is closed, $f(A)$ is not necessarily closed. Consider the Signoid function $f(x) = e^x/(1 + e^x)$ and let $A = [0, \infty)$, a closed interval. Then $f(A) = [\frac{1}{2}, 1)$, not closed.

Solution

(1) We want to show that any convergent sequence in G converges in G. Consider $\{(p_n), (f(p_n))\}$. If it converges, this implies (p_n) converges to p then the continuity of f implies $(f(p_n)) \to fp$. Hence the sequence in G converges to $(p, fp) \in G$. Then G is closed.

(2) A being compact implies A being closed and bounded. On the other hand we know, from the previous part, that G is closed. Now it suffices to show G is bounded. Yet $f(A)$ implies $f(A)$ is bounded. Hence both A bounded and $f(A)$ bounded implies G bounded.

15 Wed 10/7

Example 15.1

Let $A = \{(x, y) \in \mathbb{R}^2 \mid 1 \leqslant x^2 + y^2 \leqslant 2\}$. Does there exist a continuous function $f : A \to \mathbb{R}$ which is unbounded from above?

Solution

No. First note that A is closed and bounded in $\mathbb{R}^2$ so by Heine-Borel it is compact. Recall that a continuous function maps compact sets to compact sets, so $f(A)$ must be compact which, in turn , means it is closed and bounded.

Remark

If we define $A = \{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$ then there does exist a continuous function that blows up to infinity.

Theorem 71

If (M, d) is a metric space and $S \subset M$ is a connected subset, then anything betweem S and $\overline{S}$, i.e., any T satisfying $S \subset T \subset \overline{S}$, is connected.

Proof

We prove by contrapositive: that if T is disconnected then S is disconnected. By definition, if T is disconnected then $T = A \sqcup B$ where both A and B are proper clopen. If we define $A' = A \cap S$ and $B' = B \cap S$ then $S = A' \sqcup B'$.

Claim: A', B' are proper, nonempty, clopen subsets of S. By inheritance A' and B' are both clopen. No we'll show that A' and B' are proper and nonempty. Suppose, by contradiction, that A' is empty. Then $A = T \setminus S$. On the other hand, A is by definition open and nonempty. Hence if we pick $p \in A$, there exists $\epsilon > 0$ such that for all $x \in T, d(x,p) < \epsilon \implies x \in A$. But since $p \in \overline{S} \setminus S$, we can find a $y \in S$ satisfying $d(y,p) < \epsilon$. Then $y \in S \cap A$, contradicting the assumption that $A \subset T \setminus S$. Hence A' and B' must be nonempty, proper, and clopen. Therefore S is disconnected. Hence proven the contrapositive. □

Theorem 72: Generalized IVT

If (M,d) is a connected metric space and $f : M \to R$ a continuous function, then if $a, b \in f(M)$ and $a < b$, then for all $c \in (a,b)$ we have $c \in f(M)$.

Proof

Suppose, by contradiction, that there exists some $c \in (a,b)$ with $c \notin f(M)$. Then we can come up with a separation of M. Define $A = f^{-1}((-\infty, c))$ and $B = f^{-1}((c,\infty))$. Clearly $M = A \sqcup B$. The two sets are both nonempty. They are both open because f is continuous and thus maps open sets to open preimages. Since they are complements of each other they are also closed. Hence $A \sqcup B$ is a separation of M, contradiction M being connected. □

Theorem 73

$\mathbb{R}$ is connected.

Proof

Suppose $\mathbb{R} = A \sqcup A^c$ where A is proper clopen. By assumption A is not empty; consider $p \in A$. Consider $S = \{x \in \mathbb{R} \mid (p, x) \subset A\}$.

Claim: $(p, \infty) \subset A$. If this claim fails, then S is bounded from above and also nonempty (since A is open and we have the neighborhood argument which shows something $> p$ is also in A). If we define $c = \sup S$ we see that anything less c is contained in S and thus in A. Hence c is a limit of A. Since A is closed, $c \in A$. Again since A is open, there exists $\epsilon > 0$ with $(c - \epsilon, c + \epsilon) \in A$, contradicting $c = \sup S$. Hence $(c, \infty) \subset A$. Likewise $(-\infty, c) \subset A$. Therefore $A = \mathbb{R}$, contradicting its being clopen. Hence $\mathbb{R}$ is connected. □

Proposition 74

$(0, 1)$ is connected since it is homeomorphic to $\mathbb{R}$ (consider arctan with coefficients or the Signoid function). Since connectedness is a topological property, this interval is connected just like $\mathbb{R}$. More generally, (a, b) is also connected.

Proposition 75

$[0, 1]$ is connected: consider $f : \mathbb{R} \to [0, 1]$ defined by $x \mapsto (\sin x)/2 + 0.5$ which is continuous. Hence the image of a connected set is connected, and $[0, 1]$ is connected.

16 Fri 10/9

From last lecture, if $S \subset M$ is connected then $S \subset T \subset \overline{S} \implies T$ is connected.

Example 16.1

Let $M = \mathbb{R}^2$ and $S = \{(x, \sin(1/x) \mid x \in (0, 1]\}$, the topologist's sine curve itself. Recall that $\overline{S} = S \cup \{0\} \times [-1, 1]$[see HW for justification], and S is connected because it is a continuous image from $(0, 1] \to \mathbb{R}^2$. Then it immediately follows that

$$S \cup \{(0, x)\} \text{ for some } x \in [-1, 1]$$

is connected since this set is a supset of S and a subset of $\overline{S}$.

Definition 76: Path

Let (M, d) be a metric space. A **path** in M from $p \in M$ to $q \in M$ is a continuous function $\gamma : [0,1] \to M$ with $\gamma(p) = p$ and $\gamma(1) = q$. "The path can be drawn with a pen and the dots on the path cannot teleport."

Definition 77: Path-connectedness

M is **path-connected** if any two points in M can be joined by a path.

Proposition 78

Path connectedness is stronger than connectedness: the former implies the latter.

Proof

Look at the contrapositive. If M is disconnected, then there exists a separation $M = A \sqcup B$ where A and B are both propen clopen. Pick $a \in A$ and $b \in B$. If there exists a continuous function $f : [0,1] \to M$ creating this path, then we know $0 = f^{-1}(a) \in f^{-1}(A)$ and $1 = f^{-1}(b) \in f^{-1}(B)$, but clearly $f^{-1}(A)$ and $f^{-1}(B)$ are proper clopen subsets of $[0,1]$. (They are disjoint.) Hence $[0,1]$ is disconenected, but this is absurd. □

Remark

On the other hand, connectedness does not necessarily imply path-connectedness. Consider the closed topologist's sine curve [see HW8].

Theorem 79

Any open subset of $\mathbb{R}$ is a countable union of disjoint intervals (possibly infinite or half infinite). Examples: $(0,1)$, or $(0,1) \cup (1,2)$, and so on.

Covering Compactness

Definition 80

A **covering** of a metric space M is a collection $\mathcal{U}$ of subsets $(V_\alpha)_{\alpha \in I}$ (I being some indexing set) such that $\bigcup_{\alpha \in I} V_\alpha = M$ (note that there is no supset of M). Likewise, a **covering** of a subset $A \subset M$ is a collection of subsets $(V_\alpha)_{\alpha \in I}$ of M such that $\bigcup_{\alpha \in I} V_\alpha \supset A$.

Definition 81

An **open covering** is a covering such hat each of the subsets V_α is open.

Definition 82

A **subcovering** is a covering including a subset of the collection of subsets in the original covering.

Definition 83

A metric space M is called **covering compact** if any open open covering *has* (can be reduced to) a finite subcovering.

Theorem 84

(To be proven later) Covering compactness is equivalent to sequential compactness.

Example 16.2

(1) $(-\infty, 1) \cup (-2, 2) \cup (-1, \infty)$ is an open covering of $\mathbb{R}$. A subcovering is $(-\infty, 1) \cup (-1, \infty)$.

(2) $\mathbb{R}$ can also be covered by

$$\{(n, n+2) \mid n \in \mathbb{Z}\} = \dots \cup (0,2) \cup (1,3) \cup (2,4) \cup \dots$$

is an infinite open covering of $\mathbb{R}$. Notice that it does not have any finite subcovering! For example if we take away $(1,3)$ then 2 is no longer in the covering. Hence we've found a (one suffices) covering of $\mathbb{R}$ that has no finite subcovering. Hence $\mathbb{R}$ **is not covering compact.**

(3) $\mathbb{Z}$ is again not covering compact because

$$\mathbb{Z} = \cdots \cup \{-1\} \cup \{0\} \cup \{1\} \cup \dots$$

is an infinite open covering that has no finite subcovering.

Covering compact implies sequentially compact

Assume M is covering compact. Suppose by contradiction that it is not sequentially compact. Then we can find a sequence $(p_n) \in M$ without any convergent subsequence. Then for any $m \in M$, there exists $\epsilon > 0$ satisfying that only finitely many terms of (p_n) are inside $B_\epsilon(m)$. Now consider

$$\bigcup_{m \in M} B_\epsilon(p).$$

By covering compactness, this covering reduces to a finite covering with each subset containing finitely many points. However the sequence is infinite. By pigeonhole, a contradiction has to appear. Hence M must be sequentially compact. □

17 Mon 10/12 Nested Sets & Cantor Sets

Theorem 86: Cantor Intersection Theorem

Given a decreasing nested sequence $(M \supset C_1 \supset C_2 \supset \dots)$ of compact, nonempty subsets of a metric space (M, d), their intersection is compact and nonempty. Moreover, if the diameter $\to 0$, then the intersection is a single point.

Definition 86

The **diameter** of a set S is

$$\operatorname{diam}(A) := \sup_{x \in S, y \in S} d(x, y).$$

Remark

This can be blatantly false if the C's are not compact:

$$\bigcap_{i=1}^{\infty}(0, 1/i) = \varnothing.$$

Proof

First notice that $\bigcap C_i$ is an arbitrary union of closed sets and is therefore closed. It is also a subset of all the C_i's. Therefore $\bigcap C_i$ is compact. (Note that $\varnothing$ is also compact. Now we show that this intersection is nonempty.)

Since each C_n is nonempty, we can construct (x_n), a sequence with $x_i \in C_i$. Then clearly $(x_n) \in C_1$ and the compactness implies that it has a subsequence converging to $x \in C_1$. Then, there must also be a sub-subsequence of (x_n) that converges in C_2 since all but first terms of the original subsequence of (x_n) are guaranteed to lie in C_2, which guarantees the existence of a convergent sub-subsequence in C_2. Since limits are unique we know $x \in C_2$. Likewise, $x \in C_2$ and so on. Therefore $x \in \bigcap C_i$ and we've shown it is indeed nonempty.

For the diameter part, this is immediate by the non-emptyness of $\bigcap C_i$ and the fact that its diameter is 0, because otherwise the distance between two distinct points is positive, contradicting diameter $\to 0$. Phrased formally, suppose $\lim_{i\to\infty} \operatorname{diam}(C_i) = 0$, and $p, q \in \bigcap C_i$, if $p \neq q$, then we can pick $N \in \mathbb{N}$ such that $\operatorname{diam}(C_N) < d(p, q)$. Then

$$d(p, q) \leqslant \operatorname{diam}(C_N) < d(p, q),$$

contradiction, since $p, q \in C_N$ by definition. Therefore $p = q$. □

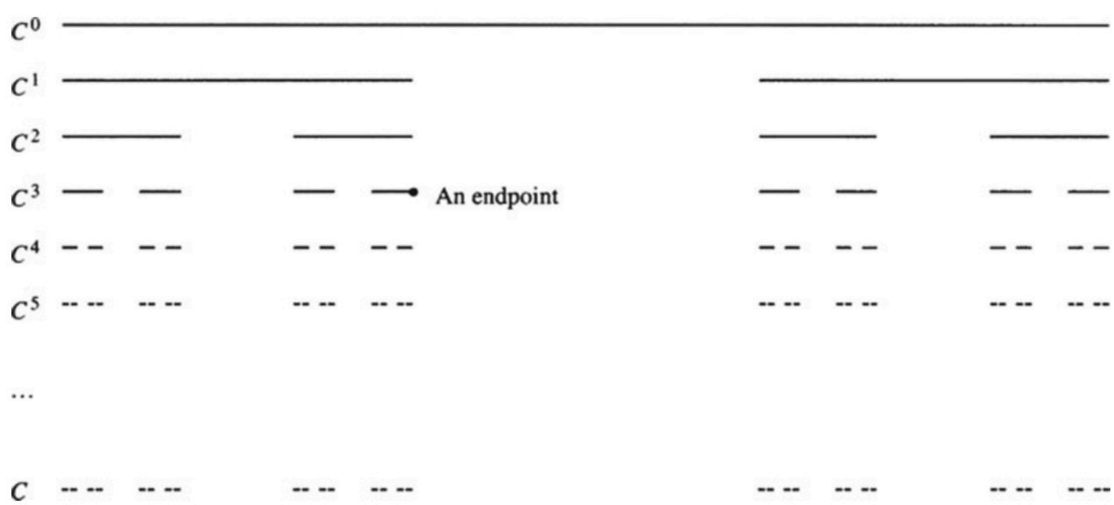
Definition 87: Standard middle thirds Cantor Set

The standard (middle-thirds) Cantor set is the intersection of the nested sequence

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C^n,$$

where $C^0 = [0,1]$ and C^{n+1} is obtained from removing the middle thirds from each interval in C^n. For example,

$$C^0 = [0,1]$$
$$C^1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$
$$C^2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$
$$\ldots$$



Lemma 17.1

$\mathcal{C}$ is nonempty: think about the fact that each C^i is compact and nonempty. Or notice that $0, 1 \in \mathcal{C}$.

Theorem 88

$\mathcal{C}$ is uncountable.

Proof

We introduce a coordinate system: let $C^1 = C_0 \cup C_2$ where $C_0 = [0, 1/3]$ and $C_2 = [2/3, 1]$. Then let the four closed intervals of C^2 be $C_{00}, C_{02}, C_{20}, C_{22}$, from left to right, respectively. Keep doing this. For example C_{220} is an interval in C^3 and also the left subinterval of C_{22}, i.e., $[0, 1/27]$. [To be finished next lecture.] ☐

18 Tue 10/13 Discussion

Recall the following concepts:

(1) Path-connected: a metric space M is said to be **path-connected** if for all $p, q \in M$ (different), there exists a continuous $f : [0,1] \to M$ with $f(0) = p, f(1) = q$.

(2) Path-connectedness, just like compactness and connectedness, is a topological property. If $f : M \to N$ is continuous and M is path-connected, then $f(M)$ is also path-connected.

Proof

Pick $p, q \in f(M)$, consider the points $f^{-1}(p), f^{-1}(q) \in M$. By the path-connectedness of M we have a continuous $\gamma : [0,1] \to M$ with $\gamma(0) = f^{-1}(p), \gamma(1) = f^{-1}(q)$.Therefore the composite $f \circ \gamma : [0,1] \to N$ satisfies $f\gamma(0) = p, f\gamma(1) = q$, and of course the composite of continuous functions is still continuous. □

(3) Path-connectedness implies connectedness but not necessarily the converse. Consider the closed topologist's sine curve.

(4) In $\mathbb{R}$, TFAE:

(I) I is path connected;

(II) I is connected;

(III) I is an interval. Definition of an interval: for all $a, b \in I$ with $a < b$, if $c \in (a, b)$ then $c \in I$.

(I) $\implies$ (II) is trivial.

For (II) $\implies$ (III), take the contrapositive. Suppose I is not an interval, then there exist $a, b \in I$ with $a < b$ such that there exists some $c \in (a, b)$ with $c \notin I$.Let $A = \{x \in I \mid x < c\}$ and $B = \{x \in I \mid x > c\}$. Note that A and B are disjoint and nonempty. Now notice that $A = (-\infty, c) \cup I$ which, by the inheritance principle, implies A is open. Likewise for B. Then taking the complements implies that A and B are clopen. Hence $A \sqcup B$ is a nontrivial separation of I and I is disconnected.

For (III) $\implies$ (II), if I is an interval, then for all $p, q \in I$, the function $f : [0,1] \to I$ defined by

$$x \mapsto (1 - x)p + xq$$

shows that I is path connected.

(5) A subset $E \subset M$ is said to be **covering compact** if for any open covering of E has a finite subcovering. For example, $(0, 1]$ is not compact because the covering

$$\bigcup_{i=1}^{\infty} (\frac{1}{i}, 2)$$

has no finite subcovering but it indeed covers $(0,1]$.

Worksheet # 9

Problem 21

Given (M,d) a metric space, and fix some $x_0 \in M$.

(1) Define a function $f : M \to \mathbb{R}$ by $f(y) = d(x_0, y)$. Show that f is continuous.

(2) Let $A \subset M$ be a compact subset and $x_0 \notin A$. Show that there exists $z \in A$ such that $d(x_0, z) \leqslant d(x_0, y)$ for all $y \in A$.

Solution

(1) Use the definition of sequential continuity: suppose $(y_n) \in M$ converges to y, then

$$d(x_0, y_n) \leqslant d(x_0, y) + d(y_n, y) \implies d(x_0, y) \leqslant d(x_0, y_n) + d(y_n, y)$$
$$d(x_0, y) \leqslant d(x_0, y_n) + d(y_n, y) \implies d(x_0, y) - d(x_0, y_n) \leqslant d(y_n, y)$$

Then we see indeed $(x_0, (y_n)) \to (x_0, y)$. Hence f is continuous.

(2) Let $h : A \to \mathbb{R}$ with $y \mapsto d(x_0, y)$ which is continuous. This is f restricted on A. Since the continuous image of a compact set is compact and therefore attains maximum and minimum, we know there exists $z \in A$ with $h(z) \leqslant h(y)$ for all $y \in A$.

Problem 22

Prove that every convex subset $E \subset \mathbb{R}^m$ is path connected.

Solution

Immediate from the definition of convexness. Suppose $p, q \in \mathbb{R}^m$ then

$$\lambda p + (1 - \lambda) p \in E.$$

Now if we define $f : [0, 1] \to E$ by

$$\lambda \mapsto \lambda p + (1 - \lambda) p$$

then this continuous function shows that E is path connected.

Problem 23

If $A \subset M$ is path connected, is $\overline{A}$ necessarily path connected?

Solution

No. Consider the topologist's sine curve and its closure.

Problem 24

Given a compact metric space (M, d), and f a function from M to N such that

$$d(f(x), f(y)) < d(x, y)$$

if $x \neq y$, prove that f has a unique fixed point in M, i.e., there exists a unique $x_0 \in M$ such that $f(x_0) = x_0$.

Solution

First we show the existence: define $h : M \to \mathbb{R}$ by $x \mapsto d(x, f(x))$. Claim: h is continuous. Suppose $(x_n) \to x$, then

$$\begin{aligned} f(x_n, f(x_n)) &\leqslant d(x_n, x) + d(x, f(x)) + d(f(x), f(x_n)) \\ d(x_n, f(x_n)) - d(x, f(x)) &\leqslant d(x_n, x) + d(f(x), f(x_n)) \\ d(x, f(x)) &\leqslant d(x, x_n) + d(x_n, f(x_n)) + d(f(x_n), f(x)) \\ d(x, f(x)) - d(x_n, f(x_n)) &\leqslant d)x_n, x) + d(f(x_n), f(x)) \end{aligned}$$

As $n \to \infty$, we have

$$|d(x, f(x)) - d(x_n, f(x_n))| \leqslant d(x_n, x) + d(f(x_n), f(x)).$$

Hence f is continuous over a compact set M. Hence there exists $x_0 \in M$ satisfying $h(x_0) = 0$. Otherwise $h(x_0) > 0$ and so $d(x_0, f(x_0)) > d(f(x_0), f(f(x_0)) = h(f(x_0))$, and this shows x_0 is no longer the minimum.

Now for the uniqueness, assume there are two x_0, y_0 with $d(x_0, y_0) = d(f(x_0), f(y_0))$. This contradicts the very assumption that LHS > RHS.

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Continuing on proving that $\mathcal{C}$ is uncountable:

Proof

(We introduce a coordinate system: let $C^1 = C_0 \cup C_2$ where $C_0 = [0, 1/3]$ and $C_2 = [2/3, 1]$. Then let the four closed intervals of C^2 be $C_{00}, C_{02}, C_{20}, C_{22}$, from left to right, respectively. Keep doing this. For example C_{220} is an interval in C^3 and also the left subinterval of C_{22}, i.e., $[0, 1/27]$.)

Define $C_{i_1 i_2 i_3 \ldots}$ as the intersection $C_{i_1} \cap C_{i_1 i_2} \cap C_{i_1 i_2 i_3} \cap \ldots$. This intersection is clearly in $\mathcal{C}$. Since each subset is compact [closed and bounded in $\mathbb{R}$], by the Cantor Intersection Theorem their intersection will be a single point. Therefore we see that there is a bijection between points $p \in \mathcal{C}$ and the set of infinite interceptions $C_{i_1 i_2 i_3 \ldots}$. Hence there exists a bijection between this set and $\mathcal{C}$. Recall $\sum_2$, the set of infinite words from an alphabet of 2 letters, which is uncountable. Hence $\mathcal{C}$ is uncountable. □

Definition 90

Let (M, d) be a metric space and let $S \subset M$ be a set. A **cluster point** of $p \in S$ is a point such that every open neighborhood of p in M contains infinitely many points of M.

Definition 90

If p is not a clutter point, it is called an **isolated point**. That is, if there exists some neighborhood that contains only p.

Remark

To show that p is a cluster point, showing that there exists *one* point in any arbitrary neighborhood of p suffices. Seeting ϵ and then setting smaller neighborhoods of radii $\epsilon/2, \epsilon/3, \ldots$ guarantees the existence of a sequence that converges to p.

Definition 91

A metric space (M, d) is said to be **perfect** if every point in M is a cluster point.

Theorem 92

$\mathcal{C}$ is perfect.

Proof

Pick $p \in \mathcal{C}$ and $\epsilon > 0$. We can *always* find a sufficiently large $n \in \mathbb{N}$ satisfying $1/3^n < \epsilon$. We do this because the interval lengths of C^n decreases by a factor of 3. This way both endpoints of the invertal of length $1/3^n$ containing p is in the ϵ-neighborhood. Therefore the neighborhood always contains some other points and p is a clustering point. □

Theorem 93

Any nonempty, perfect, complete metric space M is uncountable. [To be continued next time; Pugh p.94]

20 Fri 10/16 More on Cantor Sets

Back to the proof of theorem above: nonempty, perfect, complete implies uncountable. Notice that if M is not complete then it can be nonempty, perfect, AND countbable: think of $\mathbb{Q}$.

Proof

Suppose, by contradiction, that (M,d) is nonempty, perfect, complete, but countable (countably infinite). Then we can enumerate the points as $M = \{x_1, x_2, \dots\}$. Our goal is to find a point in M that is not in the enumeration so that we derive a contradiction.

First pick $y_1 \neq x_1 \in M$ (also in the enumeration above). Since $y_1 \neq x_1$ we can "surround y_1 with some ball whose closure excludes x_1": we can pick $1 > r_1 > 0$ satisfying $x_1 \notin \overline{B_{r_1}(y_1)}$. Since y_1 is [must be] a cluster point, there are infinitely many points of M in $B_{r_1}(y_1)$ (or its closure). Then pick $y_2 \in B_{r_1}(y_1) \subset M$ such that $1/2 > y_2 \neq x_2$ and pick $r_2 > 0$ small enough such that $x_2 \notin \overline{B_{r_2}(y_2)}$ [which is possible since $B_{r_1}(y_1)$ is open]. Then we know continue such construction since each point in M is a cluster point. Then the sequence of balls $B_{r_1}(y_1), B_{r_2}(y_2), \dots$ are a nested sequence of open balls. On the other hand, the centers, $y_1, y_2, \dots$, form a Cauchy sequence. By completeness of M this sequence converges to $y \in M$, say.

On one hand, any sequence of $(y_1, y_2, \dots)$ starting from the i'th term is enclosed entirely in the closed ball $\overline{B_{r_i}(y_i)}$. Hence the limit y must lie in all the balls.

On the other hand, $y \neq x_1$ since x_1 is excluded by the first ball; it's not x_2 since it is excluded by the second ball. More generally, it is not any x_n from the enumeration $\{x_1, x_2, \dots\}$ of M. Contradiction. Hence we cannot enumerate points in M. □

Definition 94

A subset S of a metric space (M,d) is **dense in** M if $\overline{S} = M$.

Example 20.1

$\mathbb{Q}$ is dense in $\mathbb{R}$ since $\overline{\mathbb{Q}} = \mathbb{R}$, while $\mathbb{Z}$ is not since $\overline{\mathbb{Z}} = \mathbb{Z} \neq \mathbb{R}$.

Definition 95

If S is a subset of M and U is an open subset of M, we say S is **dense in** U if $\overline{S \cap U} \supset U$.

Definition 96

A subset $S \subset M$ is **nowhere dense** if it is not in any (nonempty) open subset of M.

Theorem 97

$\mathcal{C}$ contains no intervals (a,b). Moreover, $\mathcal{C}$ is nowhere dense in $\mathbb{R}$.

Proof

Suppose $(a,b) \in \mathcal{C}$. Pick $n \in \mathbb{N}$ such that $1/3^n < b - a$. If $(a,b) \subset \mathcal{C}$, then $(a,b) \subset C^n$ which is clearly a contradiction since each interval in C^n is smaller than (a,b) itself.

Now imagine that we can find an open subset $U \in \mathbb{R}$ such that $\overline{\mathcal{C} \cap U} = U$. Then we can find an interval $(a,b) \in U$, and so

$$(a,b) \subset U \subset \overline{\mathcal{C} \cap U} \subset \overline{\mathcal{C}} = \mathcal{C},$$

contradiction. □

Definition 98

A subset $S \subset \mathbb{R}$ is a **null set** (also called a **zero set**) if, for any $\epsilon > 0$, the subset S can be expressed as a union of countably intervals (a_i, b_i) such that

$$\sum_{i=1}^{\infty} (b_i - a_i) < \epsilon.$$

(In other words, this set needs to have zero outer measure.)

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Definition 99

A metric space M is **totally disconnected** if for any $\epsilon > 0$ and any $p \in M$, we can find a clopen subset of M containing p and contained in $B_\epsilon(p)$.

Theorem 100

$\mathcal{C}$ is totally disconnected.

Proof

Take $p \in \mathcal{C}$ and any $\epsilon > 0$. We can find an interval I containing p and also contained in $(p - \epsilon, p + \epsilon)$. Now consider $U = \mathcal{C} \cap I$ — we want to show this is a clopen subset of $\mathcal{C}$.

First thing, I is a closed interval so it's closed in $\mathbb{R}$. Then the inheritance principle suggests that $U = (\mathcal{C} \cap I) \subset \mathcal{C}$ is the intersection of $\mathcal{C}$ and a closed subset of $\mathbb{R}$ and is therefore closed.

On the other hand, consider the complement of $U = \mathcal{C} \setminus U = \mathcal{C} \cap (C^n \setminus I)$, and $C^n \setminus I$ is the union of $3^n - 1$ closed intervals. Therefore by inheritance principle again, the complement of U is closed and U is open. Therefore U is clopen. □

In summary, we have shown:

Theorem 101

$\mathcal{C}$ is compact, nonempty, perfect, totally disconnected, uncountable, and has measure 0.

Definition 102

Any metric space satisfying the first 4 conditions, i.e., compact, nonempty, perfect, and totally disconnected, is called a **Cantor space**.

Theorem 103: Moore-Kline Theorem

Any two cantor spaces are homeomorphic.

Completion

Theorem 104

For every metric space (M, d), there's a complete metric $\hat{M}, \hat{d}$ such that M is a dense subset of $\hat{M}$ and d is the induced metric. Moreover, $(\hat{M}, \hat{d})$ is unique up to isometry.

Sketch of proof

Let $\hat{M}$ be the set of *all* Cauchy sequences in M. We define $\sim$ to be an equivalent relation such that, for Cauchy sequences $(p_n), (q_n)$, $(p_n) \sim (q_n)$ if $\lim_{n\to\infty} d(p_n, q_n) = 0$. Then define

$$\hat{d}((p_n), (q_n)) := \lim_{n\to\infty} d(p_n, q_n).$$

Claim: $\hat{d}$ is well-defined, i.e., independent of the preventative of equivalence class (many Cauchy sequences converge to the same limit and we can pick co-Cauchy sequences freely). Also $\hat{M}$ is complete and $M \subset \hat{M}$. $\square$

Theorem 106: Cantor Surjection Theorem

Give any compact nonempty metric space (M, d), there exists a continuous surjection $\mathcal{C} \to M$. This leads to **space filling curves**:

Theorem 106

The **Peano curve** can be written as an image of a surjective continuous map $\gamma : [0, 1] \to B^2$. Furthermore, by the Cantor surjection theorem, we can find a continuous surjective $\sigma : \mathcal{C} \to B^2$. See Pugh pg.112.

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Theorem 107: About Continuous Functions

Suppose $f : M \to N$ is continuous. The following are true:

(1) The preimage of any open $S \subset N$ under f is open.

(2) The preimage of any closed $S \subset N$ under f is closed.

(3) The image of any compact/connected/path-connected set $T \subset M$ is compact/connected/path-connected, i.e., these properties are preserved.

The following are NOT necessarily true:

(1) The continuous image of open set is open: consider $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto \sin x$. Then $f((-\infty, +\infty)) = [-1, 1]$, open to non-open.

(2) The continuous image of closed set is closed: consider the Signoid function $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto e^x/(1 + e^x)$. Then $f([0, +\infty)) = (0.5, 1)$, closed to non-closed.

(3) The preimage of a connected/path-connected set $S \subset N$ under f is connected/path-connected. Consider $f(x) : x \mapsto x^2$ and $S = [1, 4]$. Then $f^{-1}([1, 4]) = [-2, -1] \cup [1, 2]$, connected image with disconnected preimage.

(4) The continuous image of a bounded set is bounded / the preimage of complete set is complete. Consider $\tan x$ where $f((-\pi/2, \pi/2)) = \mathbb{R}$.

(5) The continuous image of a complete set is complete / the preimage of a bounded set is bounded. Similar to above, consider $\arctan x$ where $f(\mathbb{R}) = (-\pi/2, \pi/2)$.

Theorem 108: Some Classic Homeomorphisms and Non-Homeomorphisms

Classic homeomorphisms:

(1) $(0, 1) \cong \mathbb{R} : f(x) = \tan(\pi(x - 1/2))$.

(2) $S^1 \setminus \{(0, 1)\} \cong \mathbb{R}$: extend the line between $(0, 1)$ and any point on $S^1 \setminus \{(0, 1)\}$ and find its intersection with x-axis.

(3) $D_1 \cong \mathbb{R}^2$.

$$f(r, \theta) = (\tan(\frac{\pi r}{2}) \cos\theta, \tan(\frac{\pi r}{2}) \sin\theta).$$

Non-homeomorphisms:

(1) $S^1 \ncong [0, 2\pi)$. The function $(\cos\theta, \sin\theta) \mapsto \theta$ is not bicontinuous. Think of the $2\pi = 0$ issue.

(2) $S^1 \ncong \mathbb{R}^n$. For $n = 1$, remove one point from both sides. S^1 minus one point is connected but R minus one point is not. For $n \geqslant 2$, remove two points on both sides. S^2 minus two points is disconnected but $\mathbb{R}^n$ is still connected.

From Topic List

(1) Metric spaces (M, d), especially $(\mathbb{R}^n, d_{\text{st}})$ and the discrete metric space (M, d^*) which is very useful for constructing counterexamples.

  (1) If (M, d) is a finite metric space, it is disconnected — actually totally disconnected. Let $r < \inf_{x,y \in M} d_M(x, y)$ then each $p \in M$ is enclosed by some clopen neighborhood that only contains itself.

  (2) A finite metric space is also compact. By pigeonhole, if (p_n) is a sequence in M that has infinitely many terms, there's at least one $m \in M$ that appears infinitely many times in the sequence. Then the subsequence $(m, m, \dots) \to m$ and hence M has a convergent subsequence. So it's compact.

(2) To show one set is closed: show it contains all its limit points or show its complement is open. To show one set is open: show each point in it has some neighborhood contained in the set, or show its complement is closed.

(3) Definitions of continuity: let $f : M \to N$ be a function. TFAE

  (1) f is continuous if and only if it preserves sequential convergence.

  (2) f is continuous if and only if it satisfies the $\epsilon - \delta$ condition.

  (3) f is continuous if and only if the preimage of every open set $S \subset N$ under f is open (in M).

  (4) f is continuous if and only if the preimage of every closed set $S \subset N$ under f is closed (in M).

(4) Some topology $(M, \mathcal{T})$.

  (1) Finite union or arbitrary intersection of closed sets is closed. (Think of $\bigcup\{x\}$ with $x \in (0, 1)$.)

  (2) Finite intersection or arbitrary union of open sets is open. (Think of $\bigcap(1 - 1/n, 2 + 2/n), n \in \mathbb{N}$.)

(5) Given a set S we have $\text{int}(S) \subset S \subset \overline{S}$. The former is open and the latter closed.

(6) Product metrics: given M and N, $d_{\text{sum}}, d_{\text{Euclidean}}, d_{max}$ all work as metric for $M \times N$.

(7) Compact implies closed and bonded. ***Closed & bounded do not necessarily imply compact.*** Counterexample includes $\mathbb{N}$ with discrete metric. This statement is, however, true in $\mathbb{R}^n$ as guaranteed by Heine-Borel theorem.

(8) In $\mathbb{R}^n$, connected $\iff$ path-connected $\iff$ the set is an interval.

(9) Cluster point: $p \in E$ is a cluster point if (TFAE)

   (1) there exist infinitely many points within *any* neighborhood of p.

   (2) there exist one point other than p within *any* neighborhood of p.

   (3) there exist at least two points in each neighborhood of p.

   (4) there exists a sequence (not $p, p, \dots$) converging to p.

(10) Isolated point: not a cluster point. Exists some neighborhood within which there's no other point.

(11) A perfect metric space is one such that every point in it is a cluster point.

(12) Cantor set $\mathcal{C}$:

$$\mathcal{C} = \{\sum_{i=1}^{\infty} \frac{a_n}{3^n}, a_n \in \{0, 2\}\}.$$

   (1) Anything set, like $\mathcal{C}$, that is compact, nonempty, perfect, and totally disconnected, is called a Cantor space.

   (2) $\mathcal{C}$ is uncountable. It's also a null set.

22.1 Worksheet # 10

Problem 25

(1) $S^1 \ncong [0, 1)$ because if we remove one point from both sides, one become connected and the other disconnected.

(2) if $f : M \implies N$ is injective, then $f : M \to f(M)$ is bijective. True. It's injective obviously and it's surjective because everything in $f(M)$ always gets mapped to.

(3) Every subset of the discrete metric space is clopen. True.

(4) Any nonempty subset of $\mathbb{R}$ contains at least one nonempty compact subset. True. Singletons are compact in $\mathbb{R}$.

(5) In any metric space (M, d), the empty set and M are compact. False. M itself can be not compact. For example $\mathbb{R}$.

(6) If $S \subset N \subset M$ and S is open in M, then S is open in N. True by inheritance principle: $S \cap N = S$, open in N.

(7) If $A, B \subset M$. If A, B are connected then so is $A \cup B$. False. Disjoint intervals in $\mathbb{R}$ for counterexample.

(8) $\mathcal{C}$ does not contain any irrational since the endpoints of each interval is rational. False. $\mathcal{C}$ is uncountable but $\mathbb{R}$ is. It must contain some irrationals.

(9) If M is not path-connected then it's disconnected. False. Counterexample: closed topologist's sine curve.

Problem 26

Suppose (M, d) is a metric space. Let $p \in M$ and $\delta > 0$. Show that $W_p(\delta) := \{q \in M \mid d(p, q) > \delta\}$ is open in M.

Solution

The complement of this set is $\overline{B_\delta(p)}$ which is closed. Hence $W_p(\delta)$ is open. More rigorously, we want to show that for each limit point q of the complement, it's also in the complement. By the existence of a convergent sequence $(q_n) \to q$ we can find some n satisfying $d_(q_n, q) < \epsilon$ and for all later terms. Then

$$d(p, q) \leqslant d(p, q_n) + d(q_n, q) \leqslant \epsilon + \delta.$$

Problem 27

Show that every connected metric space (M, d) with at least two points is uncountable.

Solution

Suppose M is countable and has at least two points. Let $x_0 \in M$. Define $f : M \to \mathbb{R}$ by $y \mapsto d(x_0, y)$. Since f is continuous, we must have $f(M)$ a connected subset of $\mathbb{R}$, i.e., an interval. Since an interval is uncountable, so is M.