1 Fri 10/23

Definition 1

A function $f : (a, b) \to \mathbb{R}$ is **differentiable at** $c \in (a, b)$ if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists (and is finite). In this case, we put $f'(c) := $ that limit. We say f is **differentiable** if it is differentiable at *every* point in (a, b).

Lemma 1.1

If $f : (a, b) \to \mathbb{R}$ is differentiable at a point $c \in (a, b)$, then it's also continuous at c. This is because if the limit exists and finite, then the numerator, $\lim_{h\to 0}[f(c+h) - f(c)] = 0$, which is equivalent to saying f is continuous at c.

The converges is not true in general. Only **differentiable** $\implies$ **continuous** but $\not\Longleftarrow$.

Lemma 1.2

Consider $f, g : R \to R$, both of which are differentiable at $c \in \mathbb{R}$:

(1) *The derivative of sum is the sum of derivative.* $f+g$ is also differentiable at c, and $(f+g)(c) = f'(c)+g'(c)$.

(2) *Product rule:* fg is also differentiable, and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

(3) *Quotient rule:* Further assume $g(c) \neq 0$. Then f/g is differentiable at c and

$$(f/g)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.$$

(4) *Chain rule:* $(g \circ f)$ is also differentiable at c and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

Extreme Value Theorem, EVT

If $f : [a, b] \to \mathbb{R}$ is a continuous function, then there exist $c, d \in [a, b]$ such that

$$f(c) \leqslant f(x) \leqslant f(d) \text{ for all } x \in [a, b].$$

Proof

Since $[a, b]$ is compact, the continuous image is compact in $\mathbb{R}$. □

Remark

This can be blatantly false if we drop the requirement of f's continuity. Consider

$$f : [0, 1] \mapsto \mathbb{R} \text{ defined by } x \mapsto \begin{cases} k \text{ if } x = -\frac{1}{k}, k \in \mathbb{Z}_{\geqslant 1} \\ 0 \text{ otherwise.} \end{cases}$$

"Local Optimum Theorem"

If $f : (a, b) \to \mathbb{R}$ is differentiable and takes its maximal value of a point $c \in (a, b)$, then $f'(c) = 0$. Likewise for minimum.

Remark

This is again false if we take a closed interval $[a, b]$. For example consider $f(x)$ which gives $\sup f([0, 1]) = 1$ and $\inf f([0, 1]) = 0$ but the derivatives at both extrema are 1 not 0.

Proof

[Intuition: if it's positive then c is not the maximum.] Suppose, by contradiction, that $f'(c) \neq 0$. Since f is differentiable at c, there exists $h > 0$ such that

$$\left| \underbrace{\frac{f(c+h) - f(c)}{h}}_{\text{tends to } f'(c)} - f'(c) \right| < \frac{f'(c)}{2}.$$

Then

$$\frac{f(c+h) - f(c)}{h} - f'(c) > f'(c)/2 \implies f(c+h) > f(c) + \frac{f'(c)h}{2},$$

and we've found $c + h > c$ that attains a larger value than c, contradicting c's being the maximum. Likewise for other cases. $\square$

Mean Value Theorem, MVT

Given a function $f : [a, b] \to \mathbb{R}$ which is differentiable on (a, b), we can find $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

[The function is set to be defined on $[a, b]$ just to make sure $f(b), f(a)$ exist. We only care about the differentiability of f on the closed interval (a, b).]

Proof

[Intuition: draw the secant line from $f(a)$ to $f(b)$; there has to exist some point $f(c)$ on the curve with $f'(c)$ = the slope of the secant line.] Define

$$g(x) := f(x) - \frac{f(b) - f(a)}{b - a} \cdot (x - a) \text{ [eq. of secant line].}$$

First notice that $g(a) = g(b) = 0$. Recall that $g([a, b])$ is the continuous image of a compact set and must be compact. Therefore it attains a maximum and a minimum. If they both occur at the endpoints then $g(x)$ is a constant function and the derivative is the same [zero] everywhere and we can pick some $c \in (a, b)$ satisfying $g'(c) = 0$. If one of them do not occur at the endpoints then by the EVT, the one that's not the endpoint satisfy $c \in (a, b)$ and $g'(c) = 0$. Then

$$g'(c) = 0 \implies f'(c) = 0 \implies f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

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Ratio MVT

Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions that are differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

Note that we are not using fraction here to avoid dividing by 0

Proof

Consider a function $h : [a, b] \to \mathbb{R}$ defined by

$$h(x) := f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)].$$

Then

$$h'(x) = f'(x)[g(b) - g(a)] - g'(x)[f(b) - f(a)].$$

□

Here $h(a) = h(b)$ and so by MVT we claim there exists $c \in (a, b)$ with $h'(c) = 0$.

L'Hospital's Rule/Theorem

Let $f, g : (-1, 0] \to \mathbb{R}$ be differentiable functions with $g(x), g'(x) \neq 0$ for all $x \in (-1, 0)$. Suppose

$$\lim_{x \uparrow 0} f(x) = \lim_{x \uparrow 0} g(x) = 0.$$

If

$$\lim_{x \uparrow 0} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$$

then

$$\lim_{x \uparrow 0} \frac{f(x)}{g(x)} = L$$

as well.

Proof

Rough idea:

$$\frac{f(x)}{g(x)} \approx \frac{f(x) - f(t)}{g(x) - g(t)} = \frac{(f(x) - f(t))/(x - t)}{(g(x) - g(t))/(x - t)} \approx \frac{f'(x)}{g'(x)}.$$

More rigorously, by the definition of derivative, there exists some $\epsilon/2 > 0$, small enough, such that

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \frac{\epsilon}{2} \text{ for } -\frac{\epsilon}{2} < x < 0$$

Then, assuming $-\epsilon/2 < x < t < 0$, we have

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - L \right| &= \left| \frac{f(x)}{g(x)} - \frac{f(x) - f(t)}{g(x) - g(t)} + \frac{f(x) - f(t)}{g(x) - g(t)} - L \right| \\ &\leqslant \left| \frac{f(x)}{g(x)} - \frac{f(x) - f(t)}{g(x) - g(t)} \right| + \left| \frac{f(x) - f(t)}{g(x) - g(t)} - L \right| \end{aligned}$$

Note that the second absolute value is $|f'(c)/g'(c) - L|$ for some $x < c < t$ by ratio MVT. Hence that term becomes less than $\epsilon/2$ by our assumption.

Now we focus on the first term:

$$\begin{aligned} \sim &= \left| \frac{f(x)[g(x) - g(t)] - g(x)[f(x) - f(t)]}{g(x)[g(x) - g(t)]} \right| \\ &\leqslant \left| \frac{f(x)g(t)}{g(x)[g(x) - g(t)} \right| + \left| \frac{g(x)f(t)}{g(x)[g(x) - g(t)]} \right|. \end{aligned}$$

Then some approximation shows this is also $< \epsilon/2$ which finishes the proof. See Pugh p.154. $\square$

Definition 7

a function $f : (a, b) \to \mathbb{R}$ is **Lipschitz** if for some $L \in \mathbb{R}$, we always have

$$|f(x) - f(y)| \leqslant L|x - y|.$$

3 Tue 10/27 Discussion

Some definitions. Let $f : [a, b] \to \mathbb{R}$ be a function.

(1) Continuous if it satisfies the $\epsilon - \delta$ condition.

(2) *Uniformly continuous* if, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x, y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

(3) Lipschitz if there exists $L \in \mathbb{R}$ satisfying

$$|f(x) - f(y)| \leqslant L|x - y| \text{ for all } x, y.$$

(4) Differentiable if the limit exists at every point in the domain. [probably works better for open interval doamins]

(5) Hierarchy: Lipschitz $\implies$ unif. continuous $\implies$ continuous.

(6) Lipschitz does NOT imply differentiable. Consider $y = |x|$.

(7) Unif. continuous does NOT imply derivative is bounded everywhere.

Worksheet #11

Problem 1

(1) Suppose f is differentiable at point $x = a$. Show that

$$\lim_{h \to 0} \frac{f(a + h) - f(a - h)}{2h} = f'(a).$$

(2) A function f is called even if $f(x) = f(-x)$ for all x. Show that if f is even and differentiable at 0 then

$f'(0) = 0$.

Solution

(1)

$$\begin{aligned}
\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h} &= \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{2h}+\frac{f(a)-f(a-b)}{2h}\right)\\
&= \lim_{h\to 0}\frac{f(a+h)-f(a)}{2h}+\lim_{h\to 0}\frac{f(a)-f(a-h)}{2h}\\
&= \frac{1}{2}f'(a)+\frac{1}{2}f'(a)=f'(a).
\end{aligned}$$

(2) By (1),

$$\lim_{h\to 0}\frac{f(h)-f(-h)}{2h} = f'(0) = \lim_{h\to 0}\frac{f(h)-f(h)}{2h} = 0.$$

Problem 2

(1) Suppose $f:(a,b)\to\mathbb{R}$ is continuous and $f(x) > M$ for some (a,b). Show that there's a neighborhood $B_\delta(x)$ in which all point have $f(y) > M$.

(2) Show that if $f:(a,b)\to\mathbb{R}$ is continuously differentiable with $f'(x)$ unbounded, then f is not Lipschitz. Also, give an example that f is uniformly continuous.

Solution

(1) Since f is continuous at x, if we let $\epsilon = f(x) - M$, then there exists $\delta > 0$ such that

$$|y-x|<\delta \implies |f(y)-f(x)|<\epsilon \implies M < f(y).$$

(2) WLOG assume the derivative is unbounded form above. Then for each $M > 0$ there exists $x \in (a,b)$ with $f'(x) > M$. Then from (1) there exists $\delta > 0$ satisfying $y \in (x-\delta, x+\delta) \implies f'(y) > M$. Now pick y also satisfying $x < y$. Consider the interval (x,y). By MVT there exists $c \in (x,y)$ satisfying

$$M < f'(c) = \frac{f(y)-f(x)}{y-x} \implies |f(y)-f(x)| \geqslant f(y)-f(x) > M(y-x),$$

which shows f is not Lipschitz.

For a uniformly continuous with unbounded derivative, consider $f(x) = \sqrt{x}$ for $x \in (0,1)$ whose derivative is $1/2\sqrt{x}$, unbounded as $x \to 0$. To see it's uniformly continuous, extend the domain of f to $[0,1]$. Then this new function is continuous on a compact set and is uniformly continuous [to be covered soon]. Then so is the original f.

Problem 3

Suppose f is defined and differentiable for every $x > 0$ and $f'(x) \to 0$ as $x \to \infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to \infty$.

Solution

First, by MVT:

$$\frac{f(x+1) - f(x)}{(x+1) - x} = f(x+1) - f(x) = f'(y) \text{ for some } y \in (x, x+1).$$

If we let $x \to \infty$ then $f'(y) \to 0$ so $g(x) \to 0$.

4 Fri 10/30

Example 4.1

There are differentiable functions whose derivative is not continuous. Consider

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \geqslant 0 \\ 0 & x < 0 \end{cases}$$

whose derivative is

$$f'(x) = \begin{cases} 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}) & x > 0 \\ 0 & 0 \\ 0 < 0 & \end{cases}$$

This function is differentiable clearly at $x \neq 0$, and at $x = 0$ we have

$$\lim_{x \uparrow 0} = 0 = \lim_{x \downarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \downarrow 0} \frac{x^2 \sin(1/x)}{x} = 0.$$

However, the derivative oscillates violently between -1 and 1 as $x \downarrow 0$.

Definition 8

A function is **continuously differentiable** ($\in C^1$) if its differentiable and its derivative is continuous.

Definition 9

C^0 denotes the set of all continuous functions. C^1 denotes the set of all continuous functions with continuous first derivative. So on and so forth.

Theorem 10

A C^1 function $f : [a, b] \to \mathbb{R}$ is Lipschitz.

Proof

Since $f' : [a, b] \to \mathbb{R}$ is continuous, by EVT it attains maximum and minimum, i.e., there exists some L such that $|f'(x)| \leqslant L$ for all $x \in [a, b]$. Done. $\square$

Theorem 11

Let $f : (a, b) \to \mathbb{R}$ be differentiable. Then f' satisfies the intermediate value property (even for C^0 functions that are not C^1). In particular, f' cannot have a jump discontinuity.

Proof

Suppose $c < d$ for some $c, d \in (a, b)$ and pick any γ with $f'(c) < \gamma < f'(d)$. (The proof of the other side is analogous.) For $h > 0$ consider

$$S_h(x) := \frac{f(x+h) - f(x)}{h}.$$

By definition,

$$\lim_{h \to 0} S_h(c) = f'(c) \text{ and } \lim_{h \to 0} S_h(d-h) = f'(d).$$

Therefore for some n small enough, γ is sandwiched such that

$$S_h(c) < \gamma < S_h(d).$$

Notice that $S_h(x)$ is a continuous function of x. Therefore there exists some $\phi \in (c, d-h)$ such that $S_h(\phi) = \gamma$. Then

$$\frac{f(\phi+h) - f(\phi)}{h} = \gamma$$

and MVT guarantees that there indeed exists some point $\in (\phi, \phi + h) \subset (c, d)$ whose derivative is γ. $\square$

Taylor's Theorem

Suppose $f : (a, b) \to \mathbb{R}$ is r^{th} order differentiable and pick $x_0 \in (a, b)$. Define

$$P_r(x) = f(x_0) + f'(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(r)}(x_0)}{r!}(x - x_0)^r = \sum_{i=0}^{r} \frac{f^{(i)}(x - x_0)^i}{i!}.$$

By construction, this **Taylor polynomial** satisfies $P_r^i(x_0) = f^{(i)}(x_0)$ for $i = 0, 1, \ldots, r$.

Some facts:

(1) $P_r(x)$ is the "best" approximation to f near x_0 by a polynomial of degree at most r, in the sense that

$$\lim_{x \to x_0} \frac{f(x) - P_r(x)}{(x - x_0)^r} = 0 \implies P_r(x) \to f(x) \text{ quickly.}$$

(2) Moreover, $P_r(x)$ is the only polynomial that has this property.

(3) Put $R(x) := f(x) - P_r(x)$. Assume that f is actually $r+1$ differentiable. Then

$$f(x) - P_r(x) = \frac{f^{(r+1)}(\theta)(x - x_0)^r}{(r+1)!} \text{ for some } \theta \in (x_0, x).$$

If we know $|f^{(r+1)}(x)| \leqslant L$ for all x then

$$|f(x) - P_r(x)| \leqslant \frac{L}{(r+1)!}(x - x_0)^r.$$

5 Mon 11/2

Back to Taylor Approximation Theorem:

Taylor Approximation Theorem

If $f : (a,b) \to \mathbb{R}$ is r^{th} order differentiable, pick $x_0 \in (a,b)$ and define $P_r(x)$ as the following:

$$P_r(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{(x-x_0)^r}{r!} f^{(r)}(x_0),$$

then

(1) $P_r(x)$ is the "best" approximation to f near x_0 as a polynomial of degree $\leqslant r$.

(2) If f is $(r+1)^{\text{st}}$ order differentiable then if we define $R(x) = f(x) - P_r(x)$we have

$$R(x) = \frac{f^{(r+1)(\theta)}}{(r+1)!}(x-x_0)^r$$

for some θ between x_0 and x.

Proof

Not a whole proof; only proving

$$\lim_{x \to x_0} \frac{R(x)}{(x-x_0)^r} = 0.$$

By the MVT, we know that

$$\frac{R(x) - R(x_0)}{x - x_0} = R'(\theta_1) \text{ for some } \theta \in (x_0, x).$$

By construction, $P(x_0) = f(x_0), P'(x_0) = f'(x_0), \ldots, P^{(r)}(x_0) = f^{(r)}(x_0)$. Therefore $R(x_0) = R'(x_0) = \cdots = R^r(x_0) = 0$ and so $R(x) = (x - x_0)R'(\theta_1)$.

Similarly,

$$\frac{R'(\theta_1) - R'(x)}{\theta_1 - x_0} = R''(\theta_2) \text{ for some } \theta_2 \in (x_0, \theta_1).$$

Therefore $R(x) = (x - x_0)(\theta_1 - x_0)R''(\theta_2)$.

If we repeat such process, we get

$$\left| \frac{R(x)}{(x-x_0)^r} \right| \leqslant \left| \frac{R^r(\theta_r)\theta_{r-1}\ldots\theta_1(x-x_0)}{(x-x_0)^r} \right| \leqslant \left| \frac{R^r(\theta_r)}{x - x_0} \right| \to 0.$$

$\square$

Recall that, if $f : (a, b) \to (c, d)$ is strictly monotone, continuous, and surjective, then it's a homeomorphism.

Problem 4

If f is also differentiable, is its inverse f^{-1} necessarily differentiable?

Solution

No. Consider $y = x^3$. The inverse is $y = x^{1/3}$, not differentiable at the origin.

Problem 5

If $f : (a, b) \to (c, d)$ is differentiable and $f'(x)$ is never 0, is it necessarily strictly monotone?

Solution

Though f' may be discontinuous, it still satisfies the IVT [recall from last class]. Hence either f' is always positive or always negative, and from a HW problem this implies f is strictly monotone.

Inverse Function Theorem

Suppose $f : (a, b) \to (c, d)$ which is a differentiable surjection with nonzero $f'(x)$, then f^{-1} is differentiable and

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \text{ for } y \in (c, d).$$

The Riemann Integral

Definition 15

A **partition pair** $P, T \subset [a, b]$ are two finite sets of points $P = \{x_0, \ldots, x_n\}$ and $T = \{t_1, \ldots, t_n\}$, interlaced as the following:

$$a = x_0 \leqslant t_1 \leqslant x_1 \leqslant \cdots \leqslant t_n \leqslant x_n = b, x_i \neq x_j.$$

Definition 16

The **mesh** of partition P is the length of the largest subinterval.

$$\operatorname{mesh}(P) = \max_{1 \leqslant i \leqslant n} (x_i - x_{i-1}).$$

Definition 17

We define the **Riemann sum** to be

$$R(f, P, T) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

Definition 18

If $\lim_{\operatorname{mesh}(P) \to 0} R(f, P, T)$ is well-defined and finite which evaluates to $I \in \mathbb{R}$, then we say that f is Riemann integrable and define

$$\int_a^b f(x) \, \mathrm{d}x := I.$$

6 Tue 11/3 Discusion

Recall:

Theorem 19

For a differentiable function, either C^1 or not, it's always true that f' satisfies intermediate value property. In other words, f' cannot have a jump discontinuity.

Example 6.1

Suppose f is differentiable everywhere, and $f'(x)$ for $x < 0$ and $f'(x) > 3$ for $x > 0$. Show that $f'(0) = 3$.

Solution

Take $a = -1, b = 1$ say. Then $f(-1) < 3$ and $f(1) > 3$. Hence there must exist some $x \in (-1, 1)$ such that $f'(x) = 3$. This can only happen at $f'(0)$ by construction. Hence $f'(0) = 3$.

Definition 20

A function f is called r^{th}-order differentiable if $f^i(x)$ exists for all $0 \leqslant i \leqslant r$.

Example 6.2

Construct a function f which is r^{th}order differentiable but not $(r+1)^{\text{th}}$-order differentiable at x.

Solution

Consider $|x|$ which is not differential at $x = 0$.

For a function that's only first-order differentiable, consider

$$f(x) = \begin{cases} \frac{x^2}{2} & x \geqslant 0 \\ -\frac{x^2}{2} & x < 0 \end{cases}$$

and so for a r^{th}-order differentiable function, consider

$$f(x) = \begin{cases} \frac{x^{r+1}}{(r+1)!} & x \geqslant 0 \\ -\frac{x^{r+1}}{(r+1)!} & x < 0 \end{cases} \quad \text{or simply drop the factorial denominator.}$$

Recall the Taylor polynomial $P_r(h)$ of a function at x:

$$P_r(h) = \sum_{k=0}^{r} \frac{f^k(x)h^k}{k!}.$$

Taylor Approximation Theorem

Let $P_r(h)$ be the r^{th} degree Taylor polynomial of f. Then

(1) $P_r(h)$ approximates f at x in the sense that if we define $R(h) := f(x+h) - P_r(h)$ then

$$\frac{R(h)}{h^r} \to 0 \text{ as } h \to 0.$$

(2) Such $P_r(h)$ (satisfying above) of order $\leqslant r$ is unique.

(3) If f is also $(r+1)^{\text{th}}$-order differentiable, then there exists $\theta \in (x, x+h)$ such that

$$R(h) = \frac{f^{(r+1)}(\theta)}{(r+1)!} h^{r+1}.$$

Problem 6

Define

$$f(x) = |x|^3 = \begin{cases} x^3 & x \geqslant 0 \\ -x^3 & x < 0 \end{cases}$$

then write $P_2(h)$ at $x = 0$ as $P_2(h) = 0$ since $f(0) = f'(0) = f''(0) = 0$. Then $R(h) = f(x+h) = |x+h|^3$.

Definition 22

A function $f : [a, b] \to \mathbb{R}$ is called **Riemann integrable** if there exists an I such that, for each $\epsilon > 0$, there exists $\delta > 0$ such that for all P, T with $\text{mesh}(P) < \delta$ we have $|R - I| < \epsilon$.

Example 6.3

Non-example: consider Dirichlet's function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \end{cases}$$

R could be 1 if all points are rationals, and R could also be 1 if all sample points are all irrationals.

Worksheet #12

Problem 7

Suppose $f : (a, b) \to \mathbb{R}$ is differentiable at x and let $g(t) = f(t)^3$. Show directly from the definition that $g'(x) - 3xf'(x)$.

Solution

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)^3 - f(x)^3}{h} \\ &= \lim_{h\to 0} \frac{((f(x+h) - f(x))f(x+h)^2 + f(x+h)f(x) + f(x)^2}{h} \\ &= f'(x) \cdot 3f(x)^2 = 3f'(x)f^2(x). \end{aligned}$$

Problem 8

Suppose $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x) = 0$ whenever $f(x) = 0$. Prove that $|f|$ is also differentiable on (a, b).

Solution

For those x at which $f(x) \neq 0$, there exists $\delta > 0$ sufficiently small such that $f(y) > 0$ if $y \in (x-\delta, x+\delta)$. Then

$$\lim_{h\to 0} \frac{|f(x+h)| - |f(x)|}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

Analogous when $f(x) < 0$.

Now for x at which $f(x) = 0$:

$$\lim_{h\to 0} \frac{|f(x+h)| - |f(x)|}{h} = \lim_{h\to 0} \frac{|f(x+h)|}{h} \to 0 \text{ since } f(x) = 0 \implies f'(x) = 0.$$

Problem 9

In the definition of derivative, we use the difference quotient to calculate $f'(x)$. Now let us consider what happens if we take two points approaching x but neither equal to x.

(1) Suppose $f : \mathbb{R} \to \mathbb{R}$ and $f'(0)$ exists. Show that

$$\lim_{h\to 0} \frac{f(h) - f(-h)}{h - (-h)} = f'(0).$$

(2) Suppose $f : \mathbb{R} \to \mathbb{R}$ and $f'(0)$ exists, and $s_n < 0 < t_n$ with $s_n \to 0, t_n \to 0$. Show that

$$\lim_{n\to\infty} \frac{f(t_n) - f(s_n)}{t_n - s_n} = f'(0).$$

Solution

(1)

$$\begin{aligned}\lim_{h\to 0} \frac{f(h) - f(-h)}{h - (-h)} &= \lim_{h\to 0}\left(\frac{f(h) - f(0)}{2h} + \frac{f(0) - f(-h)}{2h}\right)\\ &= \frac{1}{2}\lim_{h\to 0}\frac{f(h) - f(0)}{h} + \frac{1}{2}\lim_{h\to 0}\frac{f(0) - f(-h)}{h} = f'(0).\end{aligned}$$

(2)

$$\begin{aligned}\lim_{n\to\infty} \frac{f(t_n) - f(s_n)}{t_n - s_n} &= \lim_{n\to\infty}\left(\frac{t_n(f(t_n) - f(0))}{t_n(t_n - s_n)} + \frac{-s_n(f(0) - f(s_n))}{-s_n(t_n - s_n)}\right)\\ &= \lim_{n\to\infty}\frac{t_n}{t_n - s_n}\left(\frac{f(t_n) - f(0)}{t_n} - \frac{f(0) - f(s_n)}{-s_n}\right) + \lim_{n\to\infty}\frac{f(0) - f(s_n)}{-s_n}\\ &= \lim_{n\to\infty}\frac{f(0) - f(s_n)}{-s_n} = f'(0).\end{aligned}$$

7 Wed 11/4

Definition 23

[Recall from last class] We say f is **Riemann Integrable** if, for some $I \in \mathbb{R}$, if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\operatorname{mesh}(P) < \delta \implies |R(f, P, T) - I| < \epsilon \text{ for any } P, Q.$$

In this case we define

$$\int_a^b f(\tilde{x})\, \mathrm{d}\tilde{x} = I.$$

Theorem 24

If $f : [a,b] \to \mathbb{R}$ is Riemann integrable then it's bounded. In other words, f needs to be bounded to be RI.

Proof

Suppose f is unbounded but also RI. By definition, suppose the integral evaluates to I. Take any $\epsilon > 0$. Then there exists $\delta > 0$ such that if $\text{mesh}(P) < \delta$, $|R(f,P,T) - I| < \epsilon$. Suppose

$$P = \{x_0, x_1, \ldots, x_n\} \text{ and } T = \{t_1, \ldots, t_n\}.$$

Since f is unbounded, we can assume that it's unbounded in $[x_{i-1}, x_i]$ for some i. Therefore we can find $p \in [x_{i-1}, x_i]$ such that $|f(p) - f(t_i)|(x_{i-1} - x_i)$ is sufficiently large, e.g., $> 2\epsilon$. Now consider a new partition pair

$$P' = P \text{ and } T' = \{t_1, \ldots, t_{i-1}, p, t_{i+1}, \ldots, t_n\}$$

where we replaced t_i by p. Then

$$|R(f,P',T') - R(f,P,T)| > 2\epsilon \text{ and it's impossible that both are within } [I - \epsilon, I + \epsilon].$$

Hence f cannot be RI if it's unbounded. $\square$

Remark

The converse is not true: the *Dirichlet function* is bounded but not RI. [It's Lebesgue integrable though!] Consider $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

No matter what mesh we pick, in each interval there will always be both rationals and irrationals. Hence I can evaluate to both 1 and 0 which is absurd.

Definition 25

The **lower sum** and **upper sum** of a (bounded) function $f : [a,b] \to \mathbb{R}$ with respect to a partition P of

$[a,b]$ are

$$L(f,P) = \sum_{i=1}^{n} m_i \delta x_i \text{ and } U(f,P) = \sum_{i=1}^{n} M_i \delta x_i$$

where

$$m_i = \inf\{f(t) \mid t \in [x_{i-1}, x_i]\} \text{and } M_i = \sup\{f(t) \mid t \in [x_{i-1}, x_i]\}.$$

Note that we are using inf and sup as opposed to min and max because we did *not* assume f is continuous, and it's possible that it does not attain maximum or minimum on an interval.

Definition 26

The **lower integral** and the **upper integral** of $f : [a,b] \to \mathbb{R}$ (same as above) are given by

$$\underline{I} = \sup\{L(f,P)\} \text{ and } \overline{I} = \inf\{U(f,P)\}.$$

We say f is **Darboux integrable** if $\underline{I} = \overline{I}$.

Example 7.1

The Dirichlet function is not Darboux integrable: $\underline{I} = 0$ and $\overline{I} = 1$.

Theorem 27

A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if it's Darboux integrable. In this case $\underline{I} = I = \overline{I}$.

8 Fri 11/6

Continuing from last lecture:

Theorem 28

f is Riemann integrable if and only if f is Darboux integrable. Proof: see book's "contaminated interval" proof.

Riemann Integrability Criterion

A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable (or Darboux integrable) if and only if, for each $\epsilon > 0$, there exists some partition P with $U(f, P) - L(f, P) < \epsilon$.

Refinement Principle

If we refine the partition P, i.e., add more points to it to get a new partition P' such that $P \subset P'$, then $L(f, P') \geqslant L(f, P)$ and $U(f, P') \leqslant U(f, P)$.

Theorem 31

Any function $f : [a, b] \to \mathbb{R}$ which is uniformly continuous is Riemann integrable. See proof below.

Theorem 32

Any continuous function $f : [a, b] \to \mathbb{R}$ which is continuous is uniformly continuous.

Remark

Any continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

Proof

By uniform continuity, for any $\epsilon > 0$ we can find $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b - a}.$$

Now take P a partition of $[a, b]$ with $\text{mesh}(P) < \delta$. Then $U(f, P) - L(f, P) \leqslant \epsilon$. $\square$

Theorem 33

If (M, d) is a compact metric space, then any continuous function $f : M \to \mathbb{R}$ is uniformly continuous.

Proof

Suppose not, and let $f : M \to \mathbb{R}$ be continuous but not uniformly continuous. Then there exists some $\epsilon > 0$ such that for all $\delta > 0$, there are two points, $x, y \in M$ with $d(x, y) < \delta$ but $|f(x) - f(y)| \geqslant \epsilon$.

Therefore we can find $x_1, y_1, x_2, y_2, \ldots$ such that $d(x_i, y_i) < 1/i$ while $|f(x_i) - f(y_i)| \geqslant \epsilon$. By sequential compactness, we can find a subsequence (x_{k_i}) that converges to $p \in M$. Therefore $d(y_{k_i}, p) \leqslant d(y_{k_i}, x_{k_i}) + d(x_{k_i}, p) \to 0$.

Then,

$$\epsilon \leqslant |f(x_{k_i}) - f(y_{k_i})| \leqslant |f(x_{k_i}) - f(y_{k_i})| + |f(y_{k_i}) - f(p)| \to 0$$

Contradiction. Hence f has to be uniformly continuous given M is compact. □

Riemann-Lebesgue Theorem

A function $f : [a, b] \to \mathbb{R}$ is RI if and only if it's bounded and it's continuous almost everywhere.

9 Mon 11/9

Theorem 35

Any monotone function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

Proof

WLOG assume f is monotone increasing. It suffices to show that, given$\epsilon > 0$, there exists a partition P on $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.

Let $P := \{x_0, \ldots, x_n\}$ be the partition of $[a, b]$ with each subinterval of length $\Delta x := (b - a)/n$. Then, by the monotonicity of f,

$$U(f, P) - L(f, P) = \Delta x(f(b) - f(a))$$

since, for x_i, x_{i+1}, and x_{i+2} we have

$$\sup_{x \in [x_i, x_{i+1}]} f(x) = f(x_{i+1}) = \inf_{x \in [x_{i+1}, x_{i+2}]} f(x).$$

Now if we let n sufficiently large such that $\Delta x(f(b) - f(a)) < \epsilon$ then we are done. □

Definition 36

For a point $x \in [a,b]$, we define the **oscillation** at x by

$$\operatorname{osc}_x f = \limsup_{t \to x} f(t) - \liminf_{t \to x} f(t) = \lim_{r \to 0} \operatorname{diam} f([x+r, x-r]).$$

Note that f is continuous at x if and only if $\operatorname{osc}_x f = 0$.

Example 9.1

Our favorite example: the topologist's sine curve,

$$f(x) = \begin{cases} \sin(1/x) & x > 0 \\ 0 & x \leqslant 0 \end{cases} \text{ has } \operatorname{osc}_x f(x) = 2.$$

Riemann-Lebesgue Theorem, redux

If $f : [a,b] \to \mathbb{R}$ is Riemann integrable then the set of discontinuity sets is a null set. Define

$$D := \{x \in [a,b] : f \text{ is discontinuous at } x\}$$

whereas, for $k \in \mathbb{N}$,

$$D_k = \{x \in [a,b] : \operatorname{osc}_x f \geqslant 1/k\}.$$

It's not hard to see that $D = \bigcup_{k=1}^{\infty} D_k$.

Assume $f : [a,b] \to \mathbb{R}$ is bounded and Riemann integrable. Fix $\epsilon > 0$ and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$ such that

$$U(f,P) - L(f,P) < \frac{\epsilon}{2k} \text{ for a fixed } k.$$

Therefore, if we define an interval $[x_i, x_{i+1}]$ as "bad" if (x_i, x_{i+1}) contains a point with oscillation $> 1/k$,

$$\frac{\epsilon}{2k} > \sum_{i=1}^{n} (M_i - m_i)\Delta x = \sum_{\text{bad}} (M_i - m_i)\Delta x > \frac{1}{k} \sum_{\text{bad}} \Delta x.$$

Cancellation gives $\frac{\epsilon}{2} \geqslant \sum_{\text{bad}} \Delta x$. It follows that the remaining bad points that are on the endpoints of these partitions can appear at most finitely many times. Therefore we can cover all the non-endpoint bad points by $\epsilon/2$ and the bad points on endpoints by another $\epsilon/2$. Hence D_k is a null set. Letting $k = 1, 2, \ldots$ we see that D is the countable union of null sets and is therefore a null set.

10 Tue 11/10 Discussion

Recall:

(1) The definition of Riemann integrability: $\text{mesh}(P) < \delta \implies |R - I| < \epsilon$.

(2) $1_{\mathbb{Q}\cap[0,1]}(x)$ is not Riemann integrable.

(3) If f is Riemann integrable then the Riemann integral is unique. (Use $\epsilon/2 + \epsilon/2$ trick.)

(4) If f is Riemann integrable on $[a,b]$ then $I := \int_a^b f(x)\,\mathrm{d}x = \lim_{\text{mesh}(P)\to 0} R(f,P,T)$.

(5) Linearity of integrals.

(6) Lower sum and upper sum, $L(f,P)$ and $U(f,P)$.

(7) f is Darboux integrable if $\sup L(f,P) = \inf U(f,P)$.

(8) Darboux integrable $\iff$ Riemann integrable.

(9) Definition of null sets; properties of null sets.

(10) Countable $\implies$ null set. Converse is not true: Cantor set is a null set.

Riemann Integrability Criterion

f is Riemann integrable if and only if, for each $\epsilon > 0$, there exists P such that

$$U(f,P) - L(f,P) < \epsilon.$$

Example 10.1

Define a step function

$$f(x) = \sum_{i=1}^{n} c_j 1_{[y_{i-1},y_i)} \text{ where } a = y_0 < y_1 < \cdots < b.$$

This is Riemann integrable because for each ϵ, we can simply take the partition $\{y_0, \dots\}$ that gives $U = L$.

Riemann-Lebesgue Theorem

f is Riemann integrable if and only if the set of discontinuity points of f is a null set.

Example 10.2

Every monotone function $f : [a, b] \to \mathbb{R}$ is Riemann integrable. See HW11.

Problem 10

True or false:

(1) Every function defined on $[a, b]$ is Riemann integrable. False. Take character's function of $\mathbb{Q}$.

(2) If f is Riemann integrable then so is $1/f$. False. We can have f a bump function whose reciprocal blows up.

10.1 Worksheet #13

Problem 11

Given a function $f : [0, 1] \to \mathbb{R}$ and $f(x) = 0$ a.e., prove that $f(x) = 0$ for all $x \in [0.1]$.

Solution

Since $f(x) = 0$ a.e., $D(f) = \{x \mid f(x) \neq 0\}$ is a null set. Now assume $f(x) \neq 0$. Then the continuity of f implies that all $t \in (x - \delta, x + \delta)$ satisfy $f(t) \in (f(x) - |f(x)|/2, f(x) + |f(x)|/2)$. Hence $D(f)$ is a superset of that interval and is therefore not a null set. Contradiction.

Problem 12

Prove that if f is Riemann integrable on $[a, b]$ then so is $|f|$.

Solution

Clearly $|f|$ is bounded. Now it suffices to show $D(|f|)$ is a null set.

Claim: $D(|f|) \subset D(f)$, or $D^c(|f|) \supset D^c(f)$. Pick any $x \in D^c(f)$ so f is continuous at x. If $f(x) \neq 0$ then there exists an open neighborhood of x either all positive or either all negative by continuity. If all positive then $|f| = f$; if all negative then $|f| = -f$. If $f(x) = 0$ then the $\epsilon - \delta$ continuity implies that, for all $\epsilon > 0$ there exists δ such that

$$|y - x| < \delta \implies |f(y) - f(x)| < \epsilon \implies ||f(y)| - |f(x)|| < \epsilon \text{ since the latter term is 0.}$$

Hence wherever f is continuous, $|f|$ is also continuous at that point. Hence $D(|f|) \subset D(f)$ a null set.

Remark

$|f|$ being Riemann integrable does not imply f being Riemann integrable. Consider the revised characteristic function that maps to 1 and -1 instead of 1 and 0.

Problem 13

Show that if $f \geqslant 0$ and $\int_a^b f(x)\,\mathrm{d}x = 0$ then $f(x) = 0$ a.e.

Solution

What we want to show is the same as $\{x \mid f(x) \neq 0\}$ is a null set. Claim: $\{x \mid f(x) > 0\} \subset D(f)$. Pick any t such that $f(t) > 0$ If f is continuous at t then f will map everything in $(t - \delta, t + \delta)$ to $(f(t)/2, 3f(t)/2)$. Then

$$\int_a^b f(x)\,\mathrm{d}x \geqslant \int_{t-\delta}^{t+\delta} f(x)\,\mathrm{d}x > 0, \text{ contradiction.}$$

Problem 14

Prove that:

(1) If $f, g \in \mathcal{R}[a, b]$ then so is $f + g$.

(2) If $f \in \mathcal{R}[a, b]$ and $g(x) = f(x)$ a.e. and bounded, then $g \in \mathcal{R}[a, b]$.

Solution

(1) $D(f)$ and $D(g)$ are both null sets $\implies$ so is $D(f) \cup D(g)$. We want to show $D(f + g) \subset D(f) \cup D(g)$. Take anything $x \in D(f)^c \cap D(g)^c$. Then f, g are both continuous at x and so is $f + g$. Hence $x \in D(f + g)^c$.

(2) We know that $\{x \mid g(x) \neq g(x)\}$ is a null set. Let $h(x) := g(x) - f(x)$. Then $E := \{x \mid h(x) \neq 0\}$. Also $h(x)$ is bounded. Hence $h(x) \in \mathcal{R}[a, b]$. Therefore $h(x) + f(x) = g(x) \in \mathcal{R}[a, b]$.

11 Wed 11/11

Definition 40

Let (M, d) be a metric space (not necessarily compact) and let $(U_\alpha)_{\alpha \in I}$ be an open (not necessarily countable) covering of M. Then a **Lebesgue number** for this covering is $\lambda \in \mathbb{R}_{>0}$ such that, for each $x \in M$, $B_\lambda \subset U_\alpha$ for some $\alpha \in I$.

Example 11.1

Let $\cdots \cup (-2, 0) \cup (-1, 1) \cup (0, 2) \cup \ldots$ be a covering of $\mathbb{R}$. Then the Lebesgue number is anything $\leqslant 1/2$. Usually we simply pick the largest one, in this case $\lambda = 1/2$. Note that $1/2$ happens to be half of the length of the overlaps of intervals.

To get a covering of $\mathbb{R}$ without a Lebesgue number:

$$\mathbb{R} = (-\infty, 0) \cup (-\frac{1}{2}, 1 + \frac{1}{2}) \cup (1 - \frac{1}{3}, 2 + \frac{1}{3}) \cup \cdots = (-\infty, 0) \cup \bigcup_{i=0}^{\infty} (i - \frac{1}{i+2}, i + 1 + \frac{1}{i+2}).$$

At positive integers x, the r satisfying $B_r(x) \subset U_\alpha$ gets arbitrarily small.

"Lebesgue Number lemma"

If (M, d) is compact, then every open covering has a Lebesgue number $\lambda > 0$.

Proof

Suppose some covering $(U_\alpha)_{\alpha \in I}$ does not have a Lebesgue number. Then for each $\lambda > 0$ there exists $x \in M$ such that $B_\lambda(x) \not\subset U_\alpha$ for any $\alpha \in I$. Pick $x_1, x_2, \ldots$ such that $B_{1/i}(x_i) \not\subset U_\alpha$ for any $\alpha \in I$. By sequential compactness we can find a subsequence $x_{k_1}, x_{k_2}, \ldots$ that converges to p.

Clearly $p \in U_{\alpha_0}$ for some $\alpha_i \in I$. Since U_{α_0} is open, there exists $r > 0$ such that $B_r(p) \subset U_{\alpha_0}$. Now simply pick any k_i large enough that $1/k_i < r/2$. Then the $r/2$ neighborhood of x_{k_i} is both in and not in U_{α_0}, which is absurd. $\square$

Riemann-Lebesgue Theorem (Converse)

If $f : [a, b] \to \mathbb{R}$ is bounded and the discontinuity set $D(f)$ is a null set, then f is Riemann integrable.

Proof

Recall that if we define

$$D_k := \{x \in [a, b] \mid \operatorname{osc}_x f \geqslant 1/k\} \text{ then } D = \bigcup_{k=1}^{\infty} D_k$$

Clearly if D is a null set, so is each D_k. Fix $\epsilon > 0$ arbitrary. We want to show that there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. To be continued next lecture. □

12 Fri 11/13

Continuing the proof of second part of Riemann-Lebesgue:

Proof

Let M be large enough such that $|f(x)| \leqslant M$.

Also recall that D_k is a null set, so we can cover all the points in D_k by $(a_1, b_1), (a_2, b_2), \ldots$ such that their total length $< \epsilon_1$. Meanwhile, each $x \in [a, b] \setminus D_k$ has f-variation $< 1/k$. Therefore there exists some neighborhood U_x such that

$$\sup_{x \in U_x} f(x) - \inf_{x \in U_x} f(x) < \frac{1}{k}.$$

Now notice that $(\bigcup U_x) \cup (\bigcup (a_i, b_i))$ covers $[a, b]$. Since $[a, b]$ is compact, there exists $\lambda > 0$ a Lebesgue number. Let P be a partition of $[a, b]$ with mesh $< \lambda$. Then every subinterval created by P is either wholly in some bad interval (a_i, b_I) or wholly in some good interval U_x (because of the existence of this Lebesgue number!). It follows that

$$\begin{aligned}
U - L &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\
&= \sum_{\text{bad}} (M_i - m_i)\Delta x_i + \sum_{\text{good}} (M_i - m_i)\Delta x_i \\
&< \sum_{\text{bad}} 2M\Delta x_i + \sum_{\text{good}} \frac{\Delta x_i}{k} \\
&\leqslant 2M \cdot \epsilon_1 + \frac{2(b-a)}{k}.
\end{aligned}$$

Taking $\epsilon_1 < \frac{\epsilon}{4M}$ and $k > 2(b-a)/\epsilon$ gives $U - L < \epsilon$, and so f is indeed Riemann integrable. $\square$

Corollary 43

Some corollaries:

(1) Every piecewise continuous function is Riemann integrable.

(2) The characteristic function of $S \subset [a,b]$ is Riemann integrable if and only if ∂S is a null set. Reason: the discontinuity set is ∂S. The boundary of the Cantor set is itself a null set, so the characteristic function of Cantor set is actually Riemann integrable!

Fundamental Theorem of Calculus I

Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable (not necessarily continuous). Define $F : [a,b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

Then F is continuous. Moreover, $F(x)$ is differentiable with $F'(x) = f(x)$ if f is continuous at x.

Example 12.1

Consider $f : [0,1] \to \mathbb{R}$ given by $f(x) = \begin{cases} 0 & x \in [0,1/2] \\ 1 & x \in [1/2,1] \end{cases}$. Then F is given by

$$F(x) = \begin{cases} 0 & x \in [0,\frac{1}{2}] \\ x - \frac{1}{2} & x \in [\frac{1}{2},1] \end{cases}$$

which is continuous everywhere and differentiable everywhere *except* at $x = 1/2$, precisely where f is discontinuous.

Fundamental Theorem of Calculus II

Assume $f : [a,b] \to \mathbb{R}$ is Riemann integrable. Suppose $F : [a,b] \to \mathbb{R}$ where F is differentiable *everywhere* and $F'(x) = f(x)$ for all $x \in [a,b]$. Then

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

Example 12.2

There's a function $H : [0,1] \to \mathbb{R}$ whose derivative exists and equals zero almost everywhere, but $f(0) = 0$ and $f(1) = 1$. One such example is **Devil's Staircase**.