Basics of Differentiability

(1) Differentiability: $f : (a, b) \to \mathbb{R}$ is differentiable at $x \in (a, b)$ if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is finite. We say $f'(x) := $ that limit. We say f is **differentiable** if it's differentiable at every $x \in (a, b)$.

(2) L'Hôpital's Rule: if

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

is of form $0/0$ or ∞/∞ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

(3) Extreme Value Theorem (EVT): if $f : [a, b] \to \mathbb{R}$ is a continuous function then it attains maximum and minimum at $f(c)$ and $f(d)$ for some $c, d \in [a, b]$.

(4) Mean Value Theorem (MVT): if $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) then

$$\frac{f(b) - f(a)}{b - a} = f'(c) \text{ for some } c \in (a, b).$$

Darboux's Theorem

Let $f : [a, b] \to \mathbb{R}$ be differentiable. Then f' satisfies the intermediate value property, even for C^0 functions that are not C^1. In particular, f' cannot have a jump discontinuity.

Example

The following is a classic differentiable function whose derivative is not continuous:

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x > 0 \\ 0 & \text{if } x \leqslant 0 \end{cases}$$

Lipschitz Hierarchy

(1) Lipschitz $\implies$ uniformly continuous $\implies$ continuous.

(2) Differentiable $\implies$ continuous.

(3) Lipschitz and uniformly continuous but not differentiable everywhere: $f(x) = |x|$.

(4) Differentiable but not uniformly continuous (and not Lipschitz): $f(x) = x^2$.

(5) Uniformly continuous but not Lipschitz: $f(x) = \sqrt{x}$ on $[0, 1]$.

Smoothness Classes

Definition 2

A C^r function then it's r^{th}-order differentiable with $f^{(r)}(x)$ continuous. Examples: see exercise (involving absolute values or flipping signs when $x > 0$ and $x \leqslant 0$).

Taylor Approximation Theorem

Given $f \in C^r$, we start with

$$P(h) := f(x_0) + \frac{f'(x_0)h}{1!} + \cdots + \frac{f^{(r)}(x_0)h^r}{r!}.$$

(1) $P(h)$ approximates $f(x_0)$ in the sense that

$$f(x_0 + h) - P(h) = o(h^r) \text{ or } \lim_{h \to 0} \frac{f(x_0 + h) - P(h)}{h^r} = 0.$$

(2) $P(h)$ is the only polynomial with degrees $\leqslant r$ to achieve this.

(3) If f is $(r+1)^{\text{th}}$ order differentiable then

$$R(h) = \frac{f^{(r+1)}(\theta)h^{r+1}}{(r+1)!} \text{ for some } \theta \in (x_0, x_0 + h).$$

Inverse Functions

Theorem 3

Suppose $f : (a, b) \to (c, d)$ is a differentiable surjection with $f' \neq 0$ everywhere. Then f is a homeomorphism (continuous bijection with continuous inverse). Furthermore, f^{-1} is also differentiable:

$$(f^{-1})'(y) = \frac{1}{f' \circ f^{-1}(y)}.$$

Riemann Integrals

(1) Riemann Integrability Criterion:

Theorem 4

If f is bounded, then f is Riemann integrable if and only if

given $\epsilon > 0$, there exists P such that $U(f(,P) - L(f, P) < \epsilon$.

(2) Riemann integrals are linear:

$$\int_a^b f(x) + \alpha g(x) \, \mathrm{d}x = \int_a^b f(x) \, \mathrm{d}x + \alpha \int_a^b g(x) \, \mathrm{d}x.$$

(3) Riemann integrable $\iff$ Darboux integrable.

(4) Continuous functions on $[a, b]$ are RIemann integrable.

(5) Monotone functions are also Riemann integrable.

Null Sets

(1) Every subset of a null set is also a null set.

(2) Every superset of a non-null set is non-null.

(3) Countable union of null sets is null.

(4) Uncountable union of null sets may or may not be null: $[0,1]$ and the Cantor set are both uncountable unions of singletons; the former is not a null set but the latter is.

Riemann-Lebesgue Theorem

f is Riemann integrable if and only if:

(1) f is bounded, and

(2) the discontinuity set of f is a null set.

Fundamental Theorems of Calculus

1st FTC

Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable. Then

$$F(x) := \int_a^x f(t)\,\mathrm{d}t$$

is continuous at every $x \in [a,b]$. Furthermore, if f is continuous at $x \in [a,b]$ then F is differentiable at that point.

Remark

The converse is no true in general: consider $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \neq \frac{1}{2} \\ 2 & \text{if } x = \frac{1}{2} \end{cases}.$$

Then $F(x) = \int_0^x f(t)\,\mathrm{d}t = x^2/2$ which is differentiable everywhere. We just need the "majority" of the function to be "nice" enough.

2nd FTC

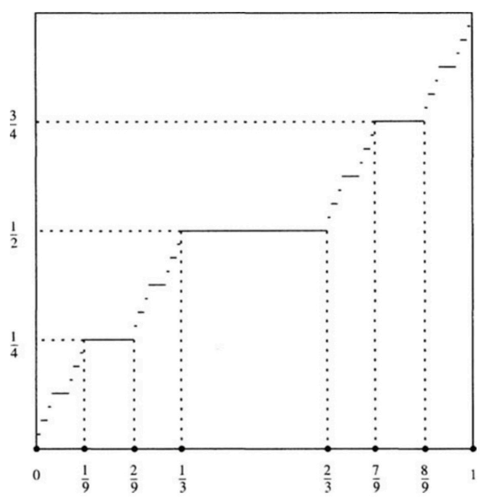
If f is differentiable on $[a, b]$, then

$$f(b) = f(a) + \int_a^b f'(t)\,dt.$$

Cantor Function / Devil's Staircase

Theorem 8

There exists a continuous and differentiable function $f : [0, 1] \to [0, 1]$ such that $f'(x) = 0$ almost everywhere but f is not a constant function.



Devil's Staircase, from Pugh's book

Worksheet: Final Reivew

Problem 1

True or false: justify.

(I) In a metric space M, if E is bounded then so is $\overline{E}$. **False**. $E \subset B_r(p) \implies \overline{E} \subset \overline{B_r(p)} \subset B_{2r}(p)$.

(II) If $E \subset M$ is open then E is not closed. **False**. Let $E := M$.

(III) The function $f(x) = x|x|$ is twice differentiable everywhere. **False**. Just false...

(IV) If $[a,b] \xrightarrow{f} [c,d] \xrightarrow{g} \mathbb{R}$ with g continuous, then $D(f) = D(g \circ f)$ [D denotes the discontinuity set]. **False**. Let g be a constant function. Then $D(g \circ f) = \varnothing$ but $D(f)$ could well be nonempty.

(V) If $Z \subset \mathbb{R}$ is a null set, then Z may be infinite. **True**. $\mathbb{Q}$.

(VI) If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and (x_n) is Cauchy, then $f(x_n)$ is also Cauchy. **True**. Can be deduced directly from the $\epsilon - \delta$ condition from uniform continuity. Once ϵ, δ are fixed, find the late enough terms in (x_n) with difference $< \delta$. Then the corresponding terms in $f(x_n)$ have difference $< \epsilon$. **Without "uniform" the statement still holds**. (x_n) Cauchy $\implies$ (x_n) convergent $\implies$ $f(x_n)$ convergent $\implies$ $f(x_n)$ Cauchy. [With the completeness of R and sequential continuity of f.

(VII) $C^\infty(\mathbb{R})$ functions are necessarily analytic on $\mathbb{R}$. **False**. Consider

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leqslant 0. \end{cases}$$

[We haven't realy talked about this.]

(VIII) If M is with discrete metric, then any mapping $f : M \to \mathbb{R}$ is uniformly continuous. **True**. Let $\delta = 1/2$ for all $\epsilon > 0$. Then

$$f(x,y) < \frac{1}{2} \implies x = y \implies d(f(x), f(y)) = 0 < \epsilon.$$

(IX) For any Riemann integrable function $f : [a,b] \to \mathbb{R}$, the function

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is differentiable at all $x \in [a,b]$. **False**. Consider

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leqslant 0 \end{cases} \quad \text{with } F(x) = |x| - 1,$$

a function not differentiable at the origin.

(X) If $f : [a,b] \to \mathbb{R}$ is monotone then f is necessarily Riemann integrable. **True**. If f is monotone then $D(f)$ is a null set. Furthermore f is bounded by $f(a)$ and $f(b)$.Hence f is Riemann integrable.

(XI) If $f : [a,b] \to \mathbb{R}$ is differentiable, then f is Riemann integrable on $[a,b]$. **True**. Differentiable on $[a,b] \implies$ contiuous on $[a,b]$. Then $f([a,b])$ is still compact $\implies$ bounded. Also, $D(f) = \varnothing$.

(XII) Given P a partition of $[a,b]$, if we refine P to Q then

$$L(f,Q) \leqslant L(f,Q) \leqslant U(f,P) \leqslant U(f,Q).$$

False. Refinement increases the lower sum and decreases the upper sum.

Problem 2

Show that if $f : [0,1] \to (0,1)$ is continuous then f is not surjective.

Solution

Immediate since continuous image of compact sets are compact. Any compact subset of (a,b) cannot be proper.

Problem 3

(1) Show that if $f : (a,b) \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing on (a,b).

Solution

MVT suggests that, for all $x < y$ we have

$$\frac{f(y) - f(x)}{y - x} = f'(c) \text{ for some } c \in (x, y).$$

Then since $f'(c) > 0$ and $y - x > 0$ we have $f(y) > f(x)$.

(2) Show that if $f : (a, b) \to \mathbb{R}$ is strictly increasing and $c \in (a, b)$, then the left and right limits both exist with

$$\lim_{x \uparrow c} f(x) \leqslant f(c) \leqslant \lim_{x \downarrow c} f(x).$$

Solution

Define

$$E := \{f(x) \mid a < x < c\} \text{ and } F := \{f(x) \mid c < x < b\}.$$

Clearly E is nonempty and is bounded from above by $f(c)$. Define $p := \sup E$. We claim the left limit of $f(x) = p$. Take a (strictly) increasing $(p_n) \to c$ with each $p_n < c$. Therefore $f(p_n) \leqslant p$ for all $n \in N$ (and also $f(p_n)$ is strictly increasing). For any $\epsilon > 0$, there exists some $x \in (a, c)$ such that

$$p - \epsilon \leqslant f(x) \leqslant p.$$

It follows that there exists $n \in \mathbb{N}$ such that $p_n > x$ (since $(p_n) \to p$ and it cannot abruptly stop increasing before even reaching x). Then for all $m \geqslant n$ we have $f(m) \geqslant f(n)$ and $|p - f(m)| < \epsilon$. Hence $\lim_{x \uparrow c} f(x) = p \leqslant f(c)$. Likewise for the other case: $\lim_{x \downarrow c} f(x) = \inf F$.

Problem 4

(1) Assume a function $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable with $f(0) = f'(0) = 0$ and $f''(0) = 2$. Show that $f(x) \geqslant 0$ in a open neighborhood of $x = 0$.

Proof

Since f is twice differentiable, we can approximate it by the Taylor polynomial:

$$\begin{aligned} P(h) &= f(0) + \frac{f'(0)h}{1!} + \frac{f''(0)h^2}{2!} \\ &= 0 + 0 + h^2 \\ &= h^2. \end{aligned}$$

Then $f(0+h) - P(h) = o(h^2)$ [where $o(h^2)$ denotes something that vanishes as $h \to 0$, for example $o(\cdot) = \cdot/h^3$]. Rearranging gives $f(h) = h^2 + o(h^2)$ and

$$\frac{f(h)}{h^2} = 1 + \frac{o(h^2)}{h^2} \to 1 \text{ as } h \to 0.$$

Therefore for sufficiently small h, $f(h) \geqslant 0$ for all $x \in (-h, h)$. □

(2) Show that the conclusion above fails if we remove the condition $f''(0) = 2$.

Solution

Consider $f(x) = x^3$.

Problem 5

Assume $f(x)$ is odd. Show by definition that if f is Riemann integrable on $[-1, 1]$ then $\int_{-1}^{1} f(x)\,\mathrm{d}x = 0$.

Solution

Assume f is Riemann integrable. Then there exists $I \in \mathbb{R}$ such that, given $\epsilon > 0$, there exists $\delta > 0$ satisfing

$$\operatorname{mesh}(P) < \delta \implies |R(f, P, T) - I| < \epsilon.$$

Let $\epsilon > 0$ be given and fix our corresponding δ. Let $n \in \mathbb{N}$ be large enough such that $1/n < \delta$. Then

$$P = \{-1, -\frac{n-1}{n}, \ldots, \frac{n-1}{n}, 1\}.$$

If we define $T \coloneqq P \setminus \{0\}$ [since we've assumed f to be Riemann intergable, it suffices to check one parititon pair P, T; if this $R(f, P, T)$ evaluates to 0 then other $R(f, P, T)$ will also evaluate to something very close to 0]. Then

$$R(f, P, T) = \sum_{k=1}^{n-1} f(-\frac{n-k}{n}) \cdot \frac{1}{n} + \sum_{i=1}^{n-1} f(\frac{n-k}{n}) \cdot \frac{1}{n} = 0 \implies |I| < \epsilon.$$

Since ϵ is arbitrary, we claim $I = 0$.