

MATH 430 Homework 0

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Problem 1. Show that $1 + 2 + \cdots + n = n(n + 1)/2$.

Solution. Suppose $\varphi(n)$ is the statement that $1 + 2 + \cdots + n = n(n + 1)/2$. We want to show that $\varphi(n)$ holds for all $n \in \mathbb{N}$ and this can be done by induction.

- (1) Base case: $n = 1$: since $1 = (1)(1 + 1)/2$, $\varphi(1)$ holds.
- (2) Inductive step: assume $\varphi(k)$ holds for some $k \in \mathbb{N}$. Then we have the identity

$$1 + 2 + \cdots + k = \frac{k(k + 1)}{2}.$$

Adding $(k + 1)$ to both sides yields

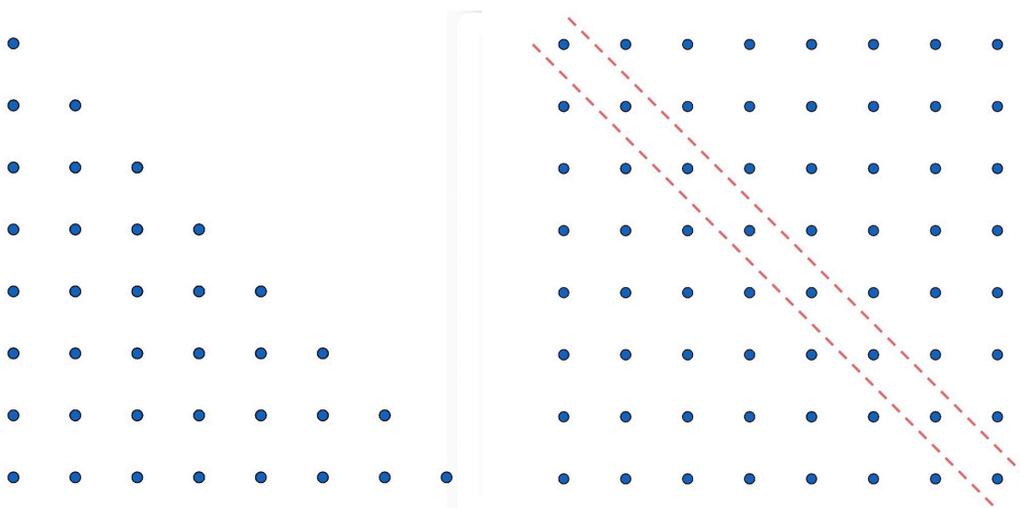
$$1 + 2 + \cdots + k + (k + 1) = \frac{k(k + 1)}{2} + \frac{2(k + 1)}{2} = \frac{(k + 1)(k + 2)}{2}$$

so $\varphi(k + 1)$ holds.

- (3) Having shown both the base case and the inductive step, we conclude that $1 + 2 + \cdots + n = n(n + 1)/2$ for all $n \in \mathbb{N}$.

Problem 2.

- (1) Draw 1 dot in the first row. Draw 2 dots in the second row. Draw 3 dots in the third row. etc. Draw a neat figure following the above process for 8 rows.
- (2) Make the triangular figure a square. How many dots are there in your completed square figure? Hence what is $2(1 + 2 + \cdots + 7) + 8$?
- (3) Hence what is $1 + 2 + \cdots + 7$?

**Solution.**

- (1) See the figure above on the left.
- (2) The completed square figure contains $8^2 = 64$ dots. Therefore $2(1 + 2 + \cdots + 7) + 8 = 64$.
- (3) It follows that $1 + 2 + \cdots + 7 = (64 - 8)/2 = 28$.

Problem 3. Do you see an alternate proof for $1 + 2 + \cdots + n = n(n + 1)/2$? Explain.

Solution. Yes... a geometric approach. The sum $1 + 2 + \cdots + n$ can be calculated by counting the number of dots in a triangular figure with n rows whose k^{th} row has k dots. If we append one more point to each row and add a first row with 1 dot all the way on top, left aligned, we now get a larger triangular figure with $(n + 1)$ rows. If we mirror this bigger triangular figure along its hypotenuse and keep both figures visible, we get a nice $(n + 1) \times (n + 1)$ square in which there are $(n + 1)^2$ dots. Since there are $(n + 1)$ dots along the diagonal, and the dots on two sides of the diagonal form two original “congruent” triangular figure with n rows (equal size & dimension), it follows that $2(1 + 2 + \cdots + n) = (n + 1)^2 - (n + 1) = n(n + 1)$. Therefore

$$1 + 2 + \cdots + n = \frac{n(n + 1)}{2}.$$