1 First Order Linear Equations and the Integrating Factor

Problem 9.1

Use an integrating factor to solve the following DEs:

(1)
$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = x^2,$$
find the general solution and the only solutions that is finite when $x = 0$.

(2)
$$\frac{\mathrm{d}x}{\mathrm{d}t} + tx = 4t$$
and find the solution with $x(0) = 2$.

(3)
$$\frac{\mathrm{d}z}{\mathrm{d}y} = z\tan y + \sin y$$
and find the general solution.

(4)
$$y' + e^{-x}y = 1$$
and find the solution when $y(0) = e$.

(5) With $a > 0$ find the solution of the equation
$$\frac{\mathrm{d}x}{\mathrm{d}t} + \left[a + \frac{1}{t}\right]x = b$$
for a general initial condition $x(1) = x_0$. Also show that $x(t) \to b/a$ as $t \to \infty$. You would get the same result if you replaced $a + 1/t$ by a.

Solution

(1) The integrating factor is $\exp(\int x^{-1}\,\mathrm{d}x) = \exp\ln x = x$. Therefore
$$I(x)y = xy = \int x^3\,\mathrm{d}x = \frac{x^4}{4} + c \implies y = \frac{x^3}{4} + \frac{c}{x}.$$
Since $c/x \to \infty$ as $x \downarrow 0$ and $c/x \to -\infty$ as $x \uparrow 0$ for all nonzero c's, the only finite solution for $x = 0$ is if $c = 0$, in which case $y = x^3/4$.

(2) Here the integrating factor is $\exp(\int t\,\mathrm{d}t) = \exp(t^2/2)$. Then

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t}e^{t^2/2} + txe^{t^2/2} = \frac{\mathrm{d}x}{\mathrm{d}t}\left[xe^{t^2/2}\right] &= 4te^{t^2/2} \\
x(t)e^{t^2/2} - x(0) &= \int_0^t 4\tilde{t}e^{\tilde{t}^2/2}\,\mathrm{d}\tilde{t} \\
x(t)e^{t^2/2} - 2 &= \left[4e^{t^2/2}\right]_{\tilde{t}=0}^t \\
x(t)e^{t^2/2} &= 4e^{t^2/2} - 2 \\
x(t) &= 4 - 2e^{-t^2/2}.
\end{aligned}$$

(3) First rewrite this DE as

$$\frac{\mathrm{d}z}{\mathrm{d}y} - (\tan y)z = \sin y.$$

Then the integrating factor is $\exp(\int -\tan y\ \mathrm{d}y) = \exp(\ln|\cos y|) = |\cos y|$. Then

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}y}\left[z|\cos y|\right] &= \sin y|\cos y| \\
z(y)|\cos y| &= \int \sin\tilde{y}|\cos\tilde{y}|\ \mathrm{d}\tilde{y} \\
z(y) &= \frac{1}{|\cos y|}\left[-\frac{\cos y|\cos y|}{2} + c\right] = \frac{c}{|\cos y|} - \frac{\cos y}{2}.
\end{aligned}$$

(4) The integrating factor is $\exp(\int e^{-x}\ \mathrm{d}x) = \exp(-e^{-x})$. Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[y\exp(-e^{-x})\right] &= \exp(-e^{-x}) \\
y(x)\exp(-e^{-x}) - y(0)\exp(-e^{-0}) &= \int_0^x \exp(-e^{-\tilde{x}})\ \mathrm{d}\tilde{x} \\
y(x)\exp(-e^{-x}) - e^{-1+1} &= \int_0^x \exp(-e^{-\tilde{x}})\ \mathrm{d}\tilde{x} \\
y(x) &= \exp(e^{-x}) + \exp(e^{-x})\int_0^x \exp(-e^{-\tilde{x}})\ \mathrm{d}\tilde{x}.
\end{aligned}$$

(5) The integrating factor is $\exp(\int(a + 1/t)\ \mathrm{d}t) = \exp(at + \ln|t|) = |t|e^{at}$. Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[x|t|e^{at}\right] &= b|t|e^{at} \\
x(t)|t|e^{at} - x(1)e^{a} &= \int_1^t b|\tilde{t}|e^{a\tilde{t}}\ \mathrm{d}\tilde{t} \\
x(t)|t|e^{at} - x_0e^{a} &= \left[\frac{b\tilde{t}e^{a\tilde{t}}\operatorname{sgn}(\tilde{t})}{a}\right]_{\tilde{t}=1}^t - \int_1^t \frac{be^{a\tilde{t}}\operatorname{sgn}(\tilde{t})}{a}\ \mathrm{d}\tilde{t} \\
&= \frac{bte^{at}\operatorname{sgn}(t) - be^{a}}{a} - \left[\frac{be^{a\tilde{t}}\operatorname{sgn}(t)}{a^2}\right]_{\tilde{t}=1}^t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{bte^{at}\operatorname{sgn}(t) - be^{a}}{a} - \frac{be^{at}\operatorname{sgn}(t) - be^{a}}{a^2} \\
x(t)|t|e^{at} &= \frac{b}{a}\left(te^{at}\operatorname{sgn}(t) - e^{a} - \frac{e^{at}\operatorname{sgn}(t) - e^{a}}{a}\right) + x_0 e^{a} \\
x(t) &= \frac{b}{a} + \frac{b}{a}\left(-\frac{e^{a(1-t)}}{|t|} - \frac{1}{at} + \frac{e^{a(1-t)}}{a|t|}\right) + \frac{x_0}{|t|}
\end{aligned}
$$

from which we see $x(t) \to b/a$ as $t \to \infty$ which then implies $|t| \to \infty$.

Problem 9.7

Suppose that

$$\frac{\mathrm{d}x}{\mathrm{d}t} \leqslant ax$$

(this is known as a differential *inequality*). Use an appropriate integrating factor and deduce that

$$x(t) \leqslant x(s)e^{a(t-s)}$$

for any t and s.

Solution

First rearrange the term and write it in the standard form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} - ax \leqslant 0.$$

Multiplying both sides by e^{-at} gives

$$\frac{\mathrm{d}x}{\mathrm{d}t}e^{-at} - axe^{-at} = \frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{-at}\right] \leqslant 0.$$

Then integrating both sides from s to t gives

$$
\begin{aligned}
\left[xe^{-a\tilde{t}}\right]_{\tilde{t}=s}^{t} &\leqslant \int_s^t 0 \, \mathrm{d}\tilde{t} \\
x(t)e^{-at} - x(s)e^{-as} &\leqslant 0 \\
x(t)e^{-at} &\leqslant x(s)e^{-as} \\
x(t) &\leqslant x(s)e^{-as+at} = x(s)e^{a(t-s)}.
\end{aligned}
$$

Problem 9.8

The function $\sin(\omega t)$ and $\cos(\omega t)$ can be written as

$$\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

and

$$\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}.$$

Use these forms to find

$$\int e^{kt} \sin(\omega t)\, \mathrm{d}t,$$

assuming that the usual rules of integration apply to such complex exponentials.

Solution

$$\begin{aligned}
\int e^{kt} \sin(\omega t)\, \mathrm{d}t &= \int e^{kt} \frac{e^{i\omega t} - e^{-i\omega t}}{2i}\, \mathrm{d}t \\
&= \int \frac{e^{(k+i\omega)t} - e^{(k-i\omega)t}}{2i}\, \mathrm{d}t \\
&= \frac{1}{2i}\left[\int e^{(k+i\omega)t}\, \mathrm{d}t - \int e^{(k-i\omega)t}\, \mathrm{d}t\right] + C \\
&= \frac{1}{2i}\left[\frac{e^{(k+i\omega)t}}{k+i\omega} - \frac{e^{(k-i\omega)t}}{k-i\omega}\right] + C \\
&= \frac{e^{kt}}{2i}\left[\frac{e^{i\omega t}}{k+i\omega} - \frac{e^{-i\omega t}}{k-i\omega}\right] + C \\
&= e^{kt}\left[\frac{(k-i\omega)e^{i\omega t} - (k+i\omega)e^{-i\omega t}}{2i(k+i\omega)(k-i\omega)}\right] + C \\
&= e^{kt}\left[\frac{k(e^{i\omega t} - e^{-i\omega t}) - i\omega(e^{i\omega t} + e^{-i\omega t})}{2i(k^2+\omega^2)}\right] + C \\
&= \frac{e^{kt}[k\sin(\omega t) - \omega\cos(\omega t)]}{k^2+\omega^2} + C.
\end{aligned}$$

2 Two 'Tricks' for Nonlinear Equations

Problem 10.1

Check that the following equations are exact and hence solve them.

(1)

$$2x - y\sec^2 x + (x^2 + 2y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

Since

$$\frac{\partial}{\partial y}(2xy - \sec^2 x) = 2x = \frac{\partial}{\partial x}(x_2^2 y)$$

we know this equation is exact. Then it has form $F(x,y) = c$. It follows that

$$F(x,y) = \int 2xy - \sec^2 x \,\mathrm{d}x = x^2 y - \tan x + C(y).$$

Since

$$\frac{\partial}{\partial y}(x^2 y - \tan x + C(y)) = x^2 + C'(y) = x^2 + 2y$$

we know $C(y) = y^2$. Therefore the solution to the original DE is

$$x^2 - \tan x + y^2 = c.$$

(2)

$$(1 + e^x y + xe^x y) + (xe^x + 2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

First check that this DE is exact:

$$\frac{\partial}{\partial y}(1 + e^x y + xe^x y) = e^x + xe^x = \frac{\partial}{\partial x}(xe^x + 2).$$

Now suppose the solution is $F(x,y) = c$, then

$$F(x,y) = \int xe^x + 2 \,\mathrm{d}y = xe^x y + 2y + C(x).$$

Differentiating with respect to x gives

$$e^x y + xe^x y + C'(x) = 1 + e^x y + xe^x y$$

and so $C'(x) = 1 \implies C(x) = x$. Therefore the solution is

$$x + xe^x y + 2y = c.$$

(3)

$$(x\cos y + \cos x)\frac{\mathrm{d}y}{\mathrm{d}x} + \sin y - y\sin x = 0.$$

Solution

First check that this equation is exact:

$$\frac{\partial}{\partial y}(\sin y - y\sin x) = \cos y - \sin x = \frac{\partial}{\partial x}(x\cos y + \cos x).$$

Therefore the solution is of form $F(x, y) = c$ and

$$F(x,y) = \int \sin y - y\sin x \,\mathrm{d}x = x\sin y + y\cos x + C(y).$$

Differentiating with respect to y gives

$$x\cos y + \cos x + C'(y) = x\cos y + \cos x \implies C'(y) = 0.$$

Hence the solution is simply

$$x\sin y + y\cos x = c.$$

(4)

$$e^x\sin y + y + (e^x\cos y + x + e^y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

First verify that this DE is exact:

$$\frac{\partial}{\partial y}(e^x\sin y + y) = e^x\cos y + 1 = \frac{\partial}{\partial x}(e^x\cos y + x + e^y).$$

Then the solution is of form $(F(x, y) = c$ and

$$F(x,y) = \int e^x\sin y + y \,\mathrm{d}x = e^x\sin y + xy + C(y).$$

Differentiating with respect to y gives

$$e^x\cos y + x + C'(y) = e^x\cos y + x + e^y$$

which suggests $C(y) = e^y$. Therefore the solution is

$$e^x\sin y + xy + e^y = c.$$

Problem 10.2

Find an integrating factor depending only on x that makes the equation

$$e^{-y}\sec x + 2\cot x - e^{-y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

exact, and hence find its solution. Hint: $\int \csc x \, \mathrm{d}x = \ln|\csc x - \cot x|$.

Solution

Suppose the integrating factor is $I(x)$, then we have

$$\begin{aligned}\frac{\partial}{\partial y}\left[I(x)(e^{-y}\sec x)\right] &= \frac{\partial}{\partial x}\left[-I(x)e^{-y}\right]\\ -I(x)e^{-y}\sec x &= -I'(x)e^{-y}\\ I(x)\sec x &= I'(x).\end{aligned}$$

Arranging the last equation as $I'(x) + \sec(x)I(x) = 0$ we see that the solution is

$$I(x) = e^{\int \sec x \, \mathrm{d}x} = \sec x + \tan x.$$

Then the original DE multiplied by $I(x)$ becomes

$$\left[(\sec x + \tan x)(e^{-y}\sec x + 2\cot x)\right] + \left[-(\sec x + \tan x)e^{-y}\right]\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

of which the solution has form $F(x, y) = C$. Then

$$\int -(\sec x + \tan x)e^{-y} \, \mathrm{d}y = (\sec x + \tan x)e^{-y} + C(x)$$

and

$$\begin{aligned}\frac{\partial}{\partial x}\left[(\sec x + \tan x)e^{-y} + C(x)\right] &= \sec x(\sec x + \tan x)e^{-y} + C'(x)\\ &= \sec x(\sec x + \tan x)e^{-y} + 2\cot x(\sec x + \tan x).\end{aligned}$$

Hence

$$\begin{aligned}C(x) &= \int 2\cot x(\sec x + \tan x) \, \mathrm{d}x\\ &= \int 2\csc x + 2 \, \mathrm{d}x\\ &= 2\ln|\csc x - \cot x| + 2 + c,\end{aligned}$$

and so the solution of the DE is given by

$$(\sec x + \tan x)e^{-y} + 2\ln|\csc x - \cot x| + 2 = C.$$

Problem 10.3

Show that any equation that can be written in the form

$$f(x) + g(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is exact, and find its solution in terms of integrals of f and g.

Solution

If $f(x)$ is a function in terms of only x and g a function in terms of only y then

$$\frac{\partial f}{\partial y} = 0 = \frac{\partial g}{\partial x}$$

which makes the DE exact. Let the solution be of form $F(x, y) = c$ and we have

$$F(x, y) = \int f(x)\,\mathrm{d}x + C(y).$$

Since

$$\frac{\partial}{\partial y}\left[\int f(x)\,\mathrm{d}x + C(y)\right] = C'(y) = g(y)$$

we know $C(y) = \int g(y)\,\mathrm{d}y$. Hence the solution is

$$\int f(x)\,\mathrm{d}x + \int g(y)\,\mathrm{d}y = c.$$

Now find the solution of

(1)

$$V'(x) + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

Let the solution be $F(x,y) = c$. Integrating $V'(x)$ with respect to x gives

$$F(x,y) = \int V'(x)\,\mathrm{d}x = V(x) + C(y)$$

whereas

$$\frac{\partial}{\partial y}[V(x) + C(y)] = 2y.$$

Therefore $C(y) = y^2$ and the solution is

$$V(x) + y^2 = c. \tag{1}$$

(2) and of

$$\left(\frac{1}{y} - a\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x} - b = 0$$

Solution

Applying the result derived from (1), the solution is

$$\int \frac{1}{y} - a\,\mathrm{d}y + \int \frac{2}{x} - b\,\mathrm{d}x = c,$$

and arranging terms gives

$$\ln y - ay + 2\ln x - bx = c.$$

Problem 10.4

By substituting $u = y/x$, solve the following homogeneous equations:

(1)

$$xy + y^2 + x^2 - x^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

If we let $u = y/x$ then $y = xu$ and $\mathrm{d}y/\mathrm{d}x = u + x\mathrm{d}u/\mathrm{d}x$. Diving the entire DE by x^2 we get

$$\frac{y}{x} + \frac{y^2}{x^2} + 1 - \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \frac{y^2}{x^2} + 1$$

which can be re-written as

$$u + x\frac{\mathrm{d}u}{\mathrm{d}x} = u + u^2 + 1 \implies x\frac{\mathrm{d}u}{\mathrm{d}x} = u^2 + 1 \implies \frac{1}{u^2+1}\mathrm{d}u = \frac{1}{x}\mathrm{d}x.$$

Integrating both sides gives

$$\int \frac{1}{u^2+1}\,\mathrm{d}u = \int \frac{1}{x}\,\mathrm{d}x \implies \arctan(u) = \arctan(y/x) = \ln x + c.$$

Thus the solution is

$$y = x\tan(\ln x + c).$$

(2)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x^2 + t\sqrt{t^2+x^2}}{tx}.$$

Solution

Again, letting $u = x/t$ we have $\mathrm{d}x/\mathrm{d}t = u + t\mathrm{d}u/\mathrm{d}t$. Now the original equation becomes

$$u + t\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{x^2 + t\sqrt{t^2+x^2}}{tx} = \frac{x}{t} + \sqrt{t^2/x^2 + 1} = u + \sqrt{1/u^2 + 1}.$$

Rearranging this equation gives

$$t\frac{\mathrm{d}u}{\mathrm{d}t} = \sqrt{1/u^2+1} \implies \frac{1}{\sqrt{1/u^2+1}}\mathrm{d}u = \frac{1}{t}\mathrm{d}t,$$

so integrating both sides and factoring out the $1/u^2$ in the LHS gives

$$\int \frac{u}{\sqrt{1+u^2}}\,\mathrm{d}u = \int \frac{1}{t}\,\mathrm{d}t \implies \sqrt{1+u^2} = \ln t + c.$$

Therefore

$$u = \pm\sqrt{(\ln t + c)^2 - 1}$$

and

$$x(t) = \pm\sqrt{(\ln t + c)^2 - 1}.$$

Problem 10.5

Solve

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kx - x^2$$

by substituting $u = x^{-1}$ and show u satisfies the linear equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 1 - ku.$$

Solve this equation for $u(t)$ and hence find the solution $x(t)$.

Solution

If we let $u = x^{-1}$ then

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -x^{-2}\frac{\mathrm{d}x}{\mathrm{d}t}$$

and hence

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= -x^{-2}(kx - x^2) \\ \implies \frac{\mathrm{d}u}{\mathrm{d}t} &= -x^{-1}k + 1 \\ \implies \frac{\mathrm{d}u}{\mathrm{d}t} &= 1 - ku \end{aligned}$$

as claimed by the question. Now we try to solve this, i.e., $\frac{\mathrm{d}u}{\mathrm{d}t} + ku = 1$. Clearly the integrating factor is e^{kt}:

$$\begin{aligned} e^{kt}\frac{\mathrm{d}u}{\mathrm{d}t} + e^{kt}ku &= e^{kt} \\ \frac{\partial}{\partial t}\left[e^{kt}u\right] &= e^{kt} \\ \int \frac{\partial}{\partial t}\left[e^{k\tilde{t}}u\right]\,\mathrm{d}\tilde{t} &= \int e^{k\tilde{t}}\,\mathrm{d}\tilde{t} \\ e^{kt}u &= \frac{1}{k}e^{kt} + c \\ u(t) &= \frac{1}{k} + ce^{-kt}. \end{aligned}$$

Substituting $x = u^{-1}$ into this equation we get the final solution:

$$x(t) = \frac{1}{1/k + ce^{-kt}}.$$

Problem 10.6

Use an appropriate substitution to solve the equation

$$\dot{x} = x(\kappa^2 - x^2).$$

Solution

Notice that this equation can be re-written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \kappa^2 x - x^3,$$

a *Bernoulli equation.* If we let $u \coloneqq x^{-2}$ then

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= -2x^{-3}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -2x^{-3}(x^3 - \kappa^2 x) \\ &= 2 + 2\kappa^2 x\end{aligned}$$

Then, for this linear equation, we can let $I(x) = \exp(\int 2\kappa^2 \, \mathrm{d}x$