Chapter 4 Graphical Representation of Solutions using MATLAB

Problem 4.1

Plot the following functions:

(1) $y(t) = \sin(5t)\sin(50t)$ for $t \in [0,3]$.

(2) $x(t) = e^{-t}(\cos(2t) + \sin(2t))$ for $t \in [0,5]$.

(3)

$$T(t) = \int_0^1 e^{-(t-x)} \sin(s)\, ds \text{ for } t \in [0,7]$$

(4) $x(t) = t \ln t$ for $t \in [0,5]$.

(5) plot y against x, where

$$x(t) = Be^{-t} + Ate^{-t} \text{ and } y(t) = Ae^{-t}$$

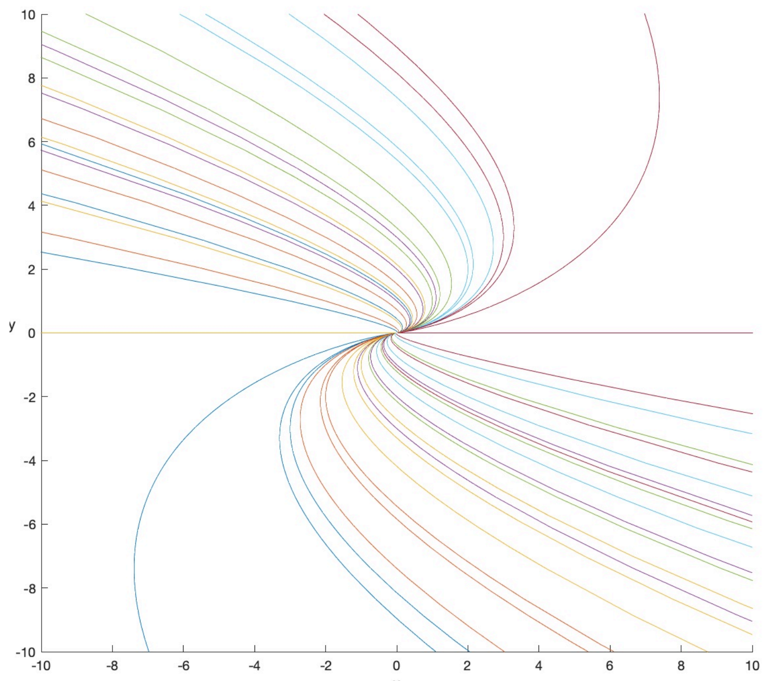
for A and B taking integer values between -3 and 3.

Solution

I will address the last part first because it is the most interesting one:

```
%part 5
hold on
for A = -3:3
    for B = -3:3
        x = @(t) B.*exp(-t) + A.*t.*exp(-t);
        y = @(t) A.*exp(-t);
        fplot(x,y);
    end
end
axis([-10,10,-10,10]);
```

A nice graph for this part:



A collection of curves defined parametrically by part (5).

For the first 4 parts, below is the MATLAB code:

```
%part 1
fplot(@(t) sin(5.*t).*sin(50.*t), [0,3]);
xlabel('t');
ylabel('y','Rotation',0);
%part 2
fplot(@(t) exp(-t).*(cos(2.*t)+sin(2.*t)), [0,5]);
xlabel('t');
ylabel('x','Rotation',0);
%part 3
fplot(@(t) integral(@(s) exp(-t+s).*sin(s), 0, t),[0,7]);
xlabel('t');
ylabel('T','Rotation',0);
%part 4
fplot(@(t) t.*log(t), [0,5]);
xlabel('t');
ylabel('x','Rotation',0);
```

Problem 4.2

Draw contour plots of the following functions:

(1) $F(x, y) = x^2 + y^2$ for $-2 \leqslant x, y \leqslant 2$.

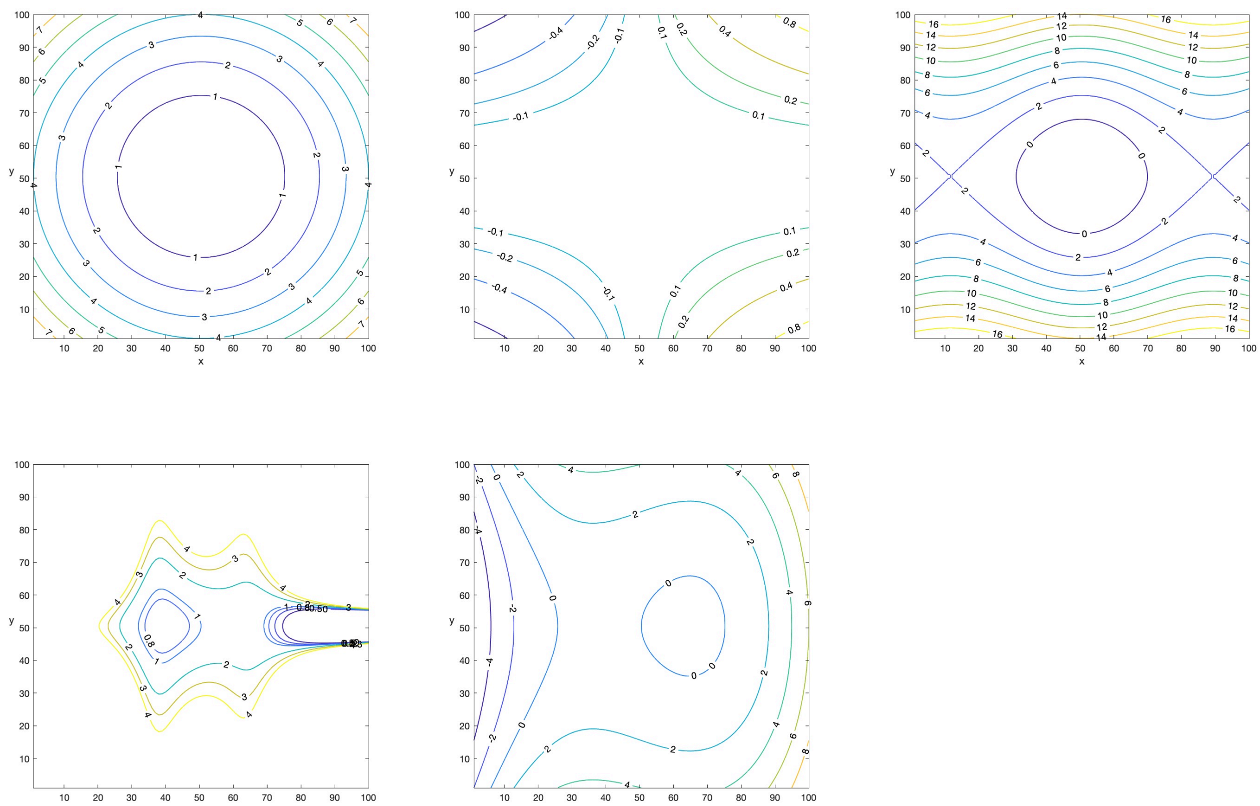
(2) $F(x, y) = xy^2$ for $-1 \leqslant x, y \leqslant 1$ with contour lines where $F = \pm 0.1, \pm 0.2, \pm 0.4, \pm 0.8$.

(3) $E(x, y) = y^2 - 2\cos x$ for $-4 \leqslant x, y \leqslant 4$.

(4) $E(x, y) = x - \frac{1}{3}x^3 + \frac{1}{2}y^2(x^4 - 2x^2 + 2)$ for $-2 \leqslant x \leqslant 4$ and $-2 \leqslant y \leqslant 2$. Draw contour lines where $E = 0, 0.5, 0.8, 1, 2, 3, 4$.

(5) $E(x, y) = y^2 + x^3 - x$ for $-2 \leqslant x, y \leqslant 2$.

Solution



The three on the first row are the first three questions; the two on the second row are the last two. Refer to the code below:

```
subplot(2,3,1); %part 1
[x1,y1] = meshgrid(linspace(-2,2), linspace(-2,2));
z1 = x1.^2 + y1.^2;
[C1, h1] = contour(z1,'LineWidth',1);
clabel(C1,h1);
xlabel('x');
ylabel('y','Rotation',0);
subplot(2,3,2); %part 2
[x2,y2] = meshgrid(linspace(-1,1), linspace(-1,1));
z2 = x2.*y2.^2;
lvl2 = ([0.1,-0.1,0.2,-0.2,0.4,-0.4,0.8,-0.8]);
[C2, h2] = contour(z2,lvl2,'LineWidth',1);
clabel(C2,h2);
xlabel('x');
ylabel('y','Rotation',0);
subplot(2,3,3); %part 3
[x3 y3] = meshgrid(linspace(-4,4), linspace(-4,4));
z3 = y3.^2-2.*cos(x3);
[C3,h3] = contour(z3, 'LineWidth', 1);
clabel(C3,h3);
xlabel('x');
ylabel('y','Rotation',0);
subplot(2,3,4); %part 4
[x4,y4] = meshgrid(linspace(-2,4), linspace(-2,2));
z4 = x4 - (1/3) .* x3.^3 + (1/2).* y3.^2 .* (x3.^4 - 2.* x3.^2 + 2);
lvl4 = ([0,0.5,0.8,1,2,3,4]);
[C4,h4] = contour(z4, lvl4, 'LineWidth', 1);
clabel(C4,h4);
xlabel('x');
ylabel('y','Rotation',0);
subplot(2,3,5); %part 5
[x5,y5] = meshgrid(linspace(-2,2), linspace(-2,2));
z5 = y5.^2 + x5.^3 -x5;
[C5,h5] = contour(z5, 'LineWidth', 1);
clabel(C5,h5);
xlabel('x');
ylabel('y','Rotation',0);
```

Chapter 5 "Trivial" Differential Equations

Problem 5.1

Find the general solution of the following equations, and in each case find the particular solution that passes through the origin.

(1)
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sin t + \cos t.$$

(2)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2 - 1} \text{ (use partial fractions.)}$$

(3)
$$\frac{\mathrm{d}U}{\mathrm{d}t} = 4t \ln t.$$

(4)
$$\frac{\mathrm{d}z}{\mathrm{d}x} = xe^{-2x}.$$

(5)
$$\frac{\mathrm{d}T}{\mathrm{d}t} = e^{-t} \sin(2t)$$

Solution

Passing through the origin $\implies$ the function $(\theta(0) = 0$.

(1)
$$\theta(t) = \int \sin t + \cos t \,\mathrm{d}t = -\cos t + \sin t + C,$$

and $\theta(0) = 0$ gives $C = 1$.

(2)
$$y(t) = \int \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) \mathrm{d}x = \frac{1}{2}(\ln|x-1| - \ln|x+1|) + C = \frac{1}{2} \ln \frac{|x-1|}{|x+1|} + C,$$

and $y(0) = 0$ gives $C = 0$.

(3)
$$U(t) = \int 4t \ln t \,\mathrm{d}x = 2t^2 \ln t - \int 2t^2 \left(\frac{1}{t}\right) \mathrm{d}t = 2t^2 \ln t - t^2 + C,$$

and $U(0)$ gives $C = 0$.

(4) ***

$$z(x) = \int xe^{-2x}\,\mathrm{d}x = -\frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}\,\mathrm{d}x = -\frac{1}{2}xe^{2x} - \frac{1}{4}e^{-2x} + C,$$

and $z(0) = 0$ gives $C = \frac{1}{4}$.

(5)

$$T(t) = \int e^{-t}\sin(2t)\,\mathrm{d}t = -e^{-t}\sin(2t) - 2\int e^{-t}\cos(2t)\,\mathrm{d}t$$

and

$$\int e^{-t}\cos(2t)\,\mathrm{d}t = -e^{-t}\cos(2t) + 2\int e^{-t}\sin(2t)\,\mathrm{d}t.$$

Then

$$\int e^{-t}\sin(2t)\,\mathrm{d}t = -e^{-t}\sin(2t) + e^{-t}\sin(2t) - 4\int e^{-t}\sin(2t)\,\mathrm{d}t$$
$$\int e^{-t}\sin(2t)\,\mathrm{d}t = \frac{e^{-t}\sin(2t) - e^{-t}\cos(2t)}{5} + C.$$

The condition $T(0) = 0$ gives $C = \frac{2}{5}$.

Problem 5.5

The *Navier-Stokes* equations that govern fluid flow were given as an example in Chapter 3. It is no possible to find explicit solutions of these equations in general. However, in certain cases, the equations reduce to something much simpler.

Suppose that a fluid is flowing down a pipe that has a circular cross-section of radius a. Assuming that the velocity V of the fluid depends only on its distance from the center of the pipe, the equation satisfied by V is

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}V}{\mathrm{d}r}\right) = -P,$$

where P is a constant.

Multiply by r and integrate once to show that

$$\frac{\mathrm{d}V}{\mathrm{d}r} = -\frac{Pr}{2} + \frac{c}{r}$$

where c is an arbitrary constant. Integrate again to find an expression of the velocity, and then use the facts

(1) the velocity should be finite at all points in the pipe and

(2) the fluids “stick” to boundaries (which means that $V(a) = 0$)

to show that

$$V(r) = \frac{P}{4}(a^2 - r^2).$$

Solution

First we integrate the equation again:

$$V = \int -\frac{Pr}{2} + \frac{c}{r}\,\mathrm{d}r = -\frac{Pr^2}{4} + c\ln r + C.$$

By the first property, we have to kill the term $c\ln r$ since $\ln r \to \infty$ as $r \to 0$, whereas V is finite for all r. Hence $c = 0$. Now by the second property we have $V(a) = 0$, which implies $C = (Pa^2)/4$. Therefore

$$V = -\frac{Pr^2}{4} + \frac{Pa^2}{4} = \frac{P}{4}(a^2 - r^2).$$

Problem 5.9

This exercise fills in the gaps in the proof of the FTC. Suppose that f is continuous at x, i.e., given any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ such that

$$|\tilde{x} - x| \leqslant \delta \implies |f(\tilde{x}) - f(x)| \leqslant \epsilon.$$

By writing

$$f(x) = \frac{1}{\delta x}\int_x^{x+\delta x} f(\tilde{x})\,\mathrm{d}\tilde{x}$$

show that for all δx with $|\delta x| \leqslant \delta$,

$$\left| f(x) - \frac{1}{\delta x}\int_x^{x+\delta x} f(\tilde{x})\,\mathrm{d}\tilde{x}\right| < \epsilon$$

and hence that

$$\lim_{\delta x\to 0}\frac{1}{\delta x}\int_x^{x+\delta x} f(\tilde{x})\mathrm{d}\tilde{x} = f(x).$$

You will need to use the fact that

$$\left|\int_a^b g(x)\,dx\mathrm{d}x\right| \leqslant \int_a^b |g(x)|\,\mathrm{d}x \leqslant (b-a)\max_{x\in[a,b]}|g(x)|.$$

Solution

Note that

$$\begin{aligned}\left| f(x) - \frac{1}{\delta x}\int_x^{x+\delta x} f(\tilde{x})\,\mathrm{d}\tilde{x}\right| &= \left|\frac{1}{\delta x}\int_x^{x+\delta x} f(x)\,\mathrm{d}x - \frac{1}{\delta x}\int_x^{x+\delta x} f(\tilde{x})\,\mathrm{d}\tilde{x}\right| \\ &\leqslant \frac{1}{\delta x}\int_x^{x+\delta x}|f(x) - f(\tilde{x})|\,\mathrm{d}\tilde{x}\end{aligned}$$

$$\leqslant \frac{1}{\delta x}\int_x^{x+\delta x}(\epsilon)\,\mathrm{d}\tilde{x}$$
$$= \frac{1}{\delta x}(\delta x)(\epsilon)$$
$$= \epsilon.$$

Chapter 6 Existence and Uniqueness of Solutions

Problem 6.1

Which of the following differential equations have unique solutions at least on some small time interval for any non-negative initial condition (($x(0) \geqslant 0$)?

(1) $\dot{x} = x(1-x^2)$

(2) $\dot{x} = x^3$

(3) $\dot{x} = x^{1/3}$

(4) $\dot{x} = \sqrt{x}(1+x)^2$

(5) $\dot{x} = (1+x)^{3/2}$

Solution

Basically the RHS needs to be Lipschitz in order to let the differential equations have unique solutions, but since the book states a stronger statement that asks the RHS to have continuous derivative:

(1) $f(x) = x(1-x^2)$ is continuous and $f'(x) = 1-3x^2$ is also continuous so the DE has unique solutions.

(2) $f(x) = x^3$ and $f'(x) = 3x^2$ are both continuous, so the DE has unique solutions.

(3) $f'(x) = \frac{1}{3x^{2/3}}$ and $f'(x) \to \infty$ as $x \to 0$. Hence the solutions are not unique.

(4) $f'(x) = -\frac{(1+x)^2}{2\sqrt{x}} + 2\sqrt{x}(1+x)$ which $\to \infty$ as $x \to 0$. Therefore the solutions are not unique.

(5) $f(x) = (1+x)^{3/2}$ and $f'(x) = \frac{3\sqrt{1+x}}{2}$ are both continuous. Hence the solutions are unique.

Problem 6.2

The Mean Value Theorem (MVT) says that if f is differentiable on an interval $[a,b]$ then $f(a) - f(b) = (b-a)f'(c)$ for some $c \in [a,b]$. Suppose that $f(x)$ is differentiable with $|f'(x)| \leqslant L$ for $a \leqslant x \leqslant b$. Use the MVT

to show that for $a \leqslant x, y \leqslant b$ we have
$$|f(x) - f(y)| \leqslant L|x - y|.$$

Solution

Clearly this is trivial when $x = y$. If $x \neq y$, WLOG assume $x > y$. Then

$$\begin{aligned}|f(x) - f(y)| &= |f(x) - f(y)| \\ &= |x - y| f'(c) \quad \text{for some } c \in [a, b] \\ &\leqslant |x - y| L.\end{aligned}$$

Problem 6.3

This exercise gives a simple proof of the uniqueness of solutions of

$$\dot{x} = f(x, t) \quad x(t_0) = x_0,$$

under the assumption that
$$|f(x, t) - f(y, t)| \leqslant L|x - y|.$$

Suppose that $x(t)$ and $y(t)$ are two solutions of the first equation. Write down the differential equation satisfied by $z(t) = x(t) - y(t)$, and hence show that

$$\frac{\mathrm{d}}{\mathrm{d}t}|z|^2 = 2z\left[f(x(t), t) - f(y(t), t)\right].$$

Now use the second equation to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}|z|^2 \leqslant 2L|z|^2.$$

If $\mathrm{d}Z/\mathrm{d}t \leqslant cZ$, it follows that $Z(t) \leqslant Z(t_0)e^{c(t-t_0)}$: use this to deduce that the solution of the first equation is unique. Hint: any two solutions agree when $t = t_0$.

Solution

If $x(t), y(t)$ are both solutions to the DE then

$$\begin{cases}\dot{x} = f(x(t), t) & x(t_0) = x_0 \\ \dot{y} = f(y(t), t) & y(t_0) = x_0\end{cases}.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}|z|^2 = 2z\dot{z} = 2z(\dot{x} - \dot{y}) = 2z[f(x(t),t) - f(y(t),t)].$$

By Lipschitz,

$$|f(x(t),t) - f(y(t),t)| \leqslant L|x(t) - y(t)| = Lz(t) \implies \frac{\mathrm{d}}{\mathrm{d}t}|z|^2 \leqslant 2zL|z| \leqslant 2L|z|^2.$$

Then, using the last property, if we set $Z = |z|^2$, then

$$|z(t)|^2 \leqslant |z(t_0)|^2 e^{2L(t-t_0)}.$$

Since $z(t_0) = x(t_0) - y(t_0) = x_0 - x_0 = 0$ but the LHS is positive (semi)definite, it follows that this inequality only holds when both sides are 0. Hence $z(t) = 0$ and $x(t) = y(t)$. The solution is unique.

Problem 6.

The proof of existence of solutions is much more involved than the proof of their uniqueness. We will consider here the slightly simpler case

$$\dot{x} = f(x) \text{ with } x(0) = x_0, \tag{1}$$

assuming that

$$|f(x) - f(y)| \leqslant L|x - y|. \tag{2}$$

The first step is to convert the differential equation into an integral equation that is easier to deal with: we integrate both sides of the first equation between 0 and t to give

$$x(t) = x_0 + \int_0^t f(x(\tilde{t})\,\mathrm{d}\tilde{t}. \tag{3}$$

This integral equation is equivalent to the original differential equation; any solution of it will solve the original one, and vice versa.

The idea behind the method is to use the RHS of (2) as a means of refining any "guess" of the solution $x_n(t)$ by replacing it with

$$x_{n+1}(t) = x_0 + \int_0^t f(x_n(\tilde{t}))\,\mathrm{d}\tilde{t}. \tag{4}$$

We start with $x_0(t) = x_0$ for all t, set

$$x_1(t) = x_0 + \int_0^t f(x_0)\,\mathrm{d}\tilde{t},$$

and continue in this way using (3). The hope is that $x_n(t)$ will converge to the solution of the differential equation as $n \to \infty$.

(1) Use (2) to show that

$$|x_{n+1}(t) - x_n(t)| \leqslant L\int_0^t |x_n(\tilde{t}) - x_{n-1}(\tilde{t})|\,\mathrm{d}\tilde{t},$$

and deduce that

$$\max_{t\in[0,1/(2L)]}|x_{n+1}(t)-x_n(t)| \leqslant \frac{1}{2}\max_{t\in[0,1/(2L)]}|x_n(t)-x_{n-1}(t)|. \tag{5}$$

Solution

The first inequality can be derived from the definitions of $x_{n+1}(t)$ and $x_n(t)$:

$$x_{n+1}(t) = x_0 + \int_0^t f(x_n(\tilde{t}))\,\mathrm{d}\tilde{t} \text{ and } x_n(t) = x_0 + \int_0^t f(x_{n-1}(\tilde{t}))\,\mathrm{d}\tilde{t}.$$

Then

$$\begin{aligned}
|x_{n+1}(t)-x_n(t)| &= \int_0^t |f(x_n(\tilde{t})) - f(x_{n-1}(\tilde{t}))|\,\mathrm{d}\tilde{t} \\
&\leqslant \int_0^t L|x_n(\tilde{t}) - x_{n-1}(\tilde{t})|\,\mathrm{d}\tilde{t} && \text{(By (2))} \\
&= L\int_0^t |x_n(\tilde{t}) - f_{n-1}(\tilde{t}))|\,\mathrm{d}\tilde{t} && \text{(First inequality proven)} \\
&\leqslant Lt \max_{\tilde{t}\in[0,t]}|x_n(\tilde{t}) - x_{n-1}(\tilde{t})|.
\end{aligned}$$

Of course, if we set $t=\frac{1}{2L}$ and therefore $[0,t]$ to be $[0,\frac{1}{2L}]$, we get

$$\max_{t\in[0,1/2L]}|x_{n+1}(t)-x_n(t)| \leqslant L\frac{1}{(2L)}\max_{\tilde{t}\in[0,1/2L]}|x_n(\tilde{t})-x_{n-1}(\tilde{t})| = \frac{1}{2}\max_{t\in[0,1/(2L)]}|x_n(t)-x_{n-1}(t)|.$$

(2) Use (5) to show that

$$\max_{t\in[0,1/(2L)]}|x_{n+1}(t)-x_n(t)| \leqslant \frac{1}{2^{n-1}}\max_{t\in[0,1/(2L)]}|x_1(t)-x_0(t)|.$$

Solution

This is immediate if we iterate (5) for n times, each time with n (of x_n) increasing by 1. We actually get a stronger statement: instead of $1/2^{N-1}$ we get $1/2^N$.

(3) By writing

$$x_n(t) = [x_n(t)-x_{n-1}(t)] + [x_{n-1}(t)-x_{n-2}(t)] + \cdots + [x_1(t)-x_0(t)] + x_0(t)$$

deduce that

$$\max_{t\in[0,1/(2L)]}|x_n(t)-x_m(t)| \leqslant \frac{1}{2^{N-2}}\max_{t\in[0,1/(2L)]}|x_1(t)-x_0(t)|$$

for all $n,m \geqslant N$.

Solution

WLOG assume $n > m$. Then

$$x_n(t)-x_m(t) = [x_n(t)-x_{n-1}(t)] + \cdots + [x_{m+1}(t)-x_m(t)]$$

and

$$\begin{aligned}\max_{t\in[0,1/(2L)]}|x_n(t)-x_m(t)| &\leqslant \left(\frac{1}{2^{n-1}}+\cdots+\frac{1}{2^m}\right)\max_{t\in[0,1/(2L)]}|x_1(t)-x_0(t)|\\ &< \frac{1}{2^{m-1}}\max_{t\in[0,1/(2L)]}|x_1(t)-x_0(t)|\\ &< \frac{1}{2^{m-2}}\max_{t\in[0,1/(2L)]}|x_1(t)-x_0(t)|\\ &\leqslant \frac{1}{2^{N-2}}\max_{t\in[0,1/(2L)]}|x_1(t)-x_0(t)|.\end{aligned}$$

It follows that $x_n(t)$ converges to some function $x_\infty(t)$ as $n\to\infty$, and therefore taking limits in both sides of (4) implies that

$$x_\infty(t) = x_0 + \int_0^t f(x_\infty(\tilde{t}))\,\mathrm{d}\tilde{t}.$$

Thus $x_\infty(t)$ satisfies (4) and is a solution of the differential equation. The previous exercise shows that the solution is unique.

Chapter 7 Scalar Autonomous ODEs

Problem 7.1/2

For each of the following differential equations, determine the stationary points and their stability analyticaslly by considering the derivative of the RHS.

(1) $\dot{x} = -x+1$

(2) $\dot{x} = x(2-x)$

(3) $\dot{x} = (1+x)(2-x)\sin x$

(4) $\dot{x} = -x(1-x)(2-x)$

(5) $\dot{x} = x^2 - x^4$

Solution

Stationary points $\iff$ $\dot{x} = 0$, and if $f(x) = 0$ then

$$\begin{cases} f'(x) > 0 \implies \text{unstable} \\ f'(x) < 0 \implies \text{stable} \\ f'(x) = 0 \implies \text{indeterminate} \end{cases}$$

(1) The stationary point is $x = 1$ and $f'(1) = 1 > 0$. Unstable.

(2) The stationary points are $x_1 = 0, x_2 = 2$. Since $f'(x) = 2 - 2x$ so $f'(x_1) > 0, f'(2) < 0$. Hence x_1 is unstable and x_2 is stable.

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}[(1+x)(2-x)\sin x] \\ &= \frac{\mathrm{d}}{\mathrm{d}x}[2\sin x + x\sin x - x^2\sin x] \\ &= 2\cos x + \sin x + x\cos x - x^2\cos x - 2x\sin x \\ &= (1-2x)\sin x + (2+x-x^2)\cos x \end{aligned}$$

we have

$$\begin{cases} f'(x_1) = 3\sin(-1) < 0 \implies \text{stable} \\ f'(x_2) = (-3)\sin(2) < 0 \implies \text{stable} \\ f'(x^*) = 2 + x - x^2 = -(x+1)(x-2)\cos(x) \begin{cases} > 0 \implies \text{unstable} \\ < 0 \implies \text{stable} \end{cases} \end{cases}$$

(3) The stationary points are $x_1 = 0, x_2 = 1$, and $x_3 = 2$. Then since $f(x) = -x(1-x)(2-x) = -2x + 3x^2 - x^3$, we know $f'(x) = -2 + 6x - 3x^2$. Then $f'(x_1) = -2 < 0$ implies x_1 is stable; $f'(x_2) = 1 > 0$ implies x_2 is unstable; $f'(x_3) = -2 < 0$ implies x_3 is stable.

(4) The stationary points are when $x^2 = x^4 \implies x_1 = 0, x_2 = -1, x_3 = 1$. Since $f'(x) = 2x - 4x^3$, we know $f'(x_1) = 0, f'(x_2) = -6 < 0, f'(x_3) = -2 < 0$. We know x_2, x_3 are stable, but no conclusion can be drawn about x_1 simply from looking at the derivative. However, the phase diagram suggests that f has different signs on different sides of 0 which implies x_1 is semi-stable.

Problem 7.4

A simple model of the spread of an infection in a population is

$$\dot{H} = -kIH \text{ and } \dot{I} = kIH,$$

where $H(t)$ is the number of healthy people, $I(t)$ is the number of infected people and k the rate of infection. Since $(\mathrm{d}/\mathrm{d}t)(H + I) = 0$, it follows that the size of the population is constant, $H + I = N$, say. Substitute $I = N - H$ in order to obtain a single equation for $H(t)$,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -kH(N - H).$$

Determine the stability of stationary points for this equation, and draw its phase diagram. Deduce that eventually all the population becomes infected.

Solution

Treating k and N as constants we find that the only stationary points are $H_1 = 0$ and $H_2 = N$. Note that the derivative of the RHS is $-kN + 2kH$, so H_1 (when everybody is infected) is stable and H_2 (when nobody is infected) is unstable.

Problem 7.8

Assume $f \in C^1$. Let $x(t)$ be one solution of the differential equation

$$\dot{x} = f(x).$$

Show that

(1) If $f(x(t^*)) = 0$ for some t^* then $x(t) = x(t^*)$ for all $t \in \mathbb{R}$ (the solution is constant, and $x(t^*)$ is a stationary point); and hence

(2) If $f(x(t^*)) > 0$ for some t^* then $f(x(t)) > 0$ for all $t \in \mathbb{R}$ (the solution cannot "reverse direction"). Hint: use the IVT: if g is a continuous function with $g(a) < 0$ and $g(b) > 0$ then there is a point c between a, b with $g(c) = 0$.

Of course, a similar result to (2) holds if $f(x(t^*)) < 0$ for some t^*.

Solution

(1) Note that $x(t) = x(t^*) = 0$, a constant function, is indeed a solution to $\dot{x} = f(x)$. Now since $f(x) \in C^1$, we know it must have a unique solution. Therefore it is exactly the one we have just stated.

(2) Suppose, by contradiction, that $f(x(t^*)) > 0$ but there exists $f(x(t')) < 0$. Then by the IVT there exists some $s \in (t^*, t')$ satisfying $f(x(s)) = 0$. By (1) this implies f is the constant function $f(x) = 0$, contradicting $f(x(t^*)) > 0$. Hence if $f(x(t^*)) > 0$ then the function is positive definite. Likewise, if $f(x(t^*)) < 0$ then the function is *always* negative.

Problem 7.9

Show that for autonomous scalar equations, if x^* is attracting then it must also be stable. Hint: use (2) above.

Solution

By the definition of attracting points, if x^* is attracting then there exists a $\delta > 0$ satisfying

$$|x_0 - x^*| < \delta \implies x(t) \to x^* \text{ as } t \to +\infty.$$

WLOG assume $x_0 < x^*$. Then the function is increasing at least at some $f(x(t^*))$. By (2) from the previous problem we then know f is strictly increasing. Therefore

$$|x(0) - x^*| < \delta \to |x(t) - x^*| < \delta \text{ for all } t \geqslant 0.$$

This implies that, if $x_0 < x^*$, given ϵ we can simply choose $\delta = \epsilon$ and the condition holds. Likewise for the case when $x_0 > x^*$.

Problem 7.10

Suppose that $x(t)$ is a solution of $\dot{x} = f(x)$ that is moving to the right. Show that either $x(t) \to +\infty$ or $x(t) \to x^*$ where x^* is a stationary point. Hint: if $x(t)$ does not tend to infinity then it is increasing and bounded above, and so it tends to a limit x^*. Show that in this case we must have $f(x^*) = 0$. A similar result holds if $x(t)$ is moving to the left, with $+\infty$ replaced by $-\infty$.

Solution

If it is the latter case (that $x(t) \nrightarrow +\infty$ then it tends to some x^*. We will now show that $f(x^*) = 0$. Suppose not, then either $f(x^*) < 0$ or > 0. If it is negative, then at some point x_1 sufficiently close to x^* with $x_1 < x^*$ we have $f(x_1) < 0$, and this contradicts $f(x)$'s moving to the right. On the other hand, if $f(x^*) > 0$, by the continuity of f, we know that there exists an ϵ-neighborhood of x^* satisfying

$$\boxed{\text{If } x \in (x^* - \epsilon, x^* + \epsilon), \text{ then } f(x) \in (f(x^*) - 0.5f(x^*), f(x^*) + 0.5f(x^*)).}$$

We will only focus on the $f(x) > f(x^*)/2$ part, namely that there exists $\epsilon > 0$ satisfying that $f(x) > 0.5f(x^*)$ for all $x \in (x^* - \epsilon, x^*)$. Then, since $x(t) \to x^*$, there exists a sufficiently large T_1 satisfying $x^* - \epsilon < x(T) < x^*$. Notice that, for $x(t) \in (x^* - \epsilon, x^*)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = f(x(t)) > \frac{1}{2} f(x^*) > 0.$$

$$\begin{aligned}
& & \int_{T_1}^{T_2} f(x(\tilde{t}))\,\mathrm{d}\tilde{t} &> \int_{T_1}^{T_2} \frac{1}{2} f(x^*)\,\mathrm{d}\tilde{t} \\
\Longrightarrow & & x(T_2) - x(T_1) &> \frac{1}{2}(T_2 - T_1) f(x^*) \\
\Longrightarrow & & x(T_2) &> x(T_1) + \frac{1}{2}(T_2 - T_1) f(x^*) \\
& & &> x^* - \epsilon + \frac{1}{2}(T_2 - T_1) f(x^*).
\end{aligned}$$

Notice that there is no restriction on how large T_2 can be. If we set

$$T_2 \geqslant T_1 + \frac{2\epsilon}{f(x^*)}$$

then

$$x(T_2) > x^* - \epsilon + \frac{1}{2}\left(\frac{2\epsilon}{f(x^*)}\right) f(x^*) = x^*,$$

contradicting $x(t)$ strictly increasing and converging to x^*. Hence $f(x^*)$ must be 0.

Chapter 8 Separable Equations

Problem 8/1 8.1(e)

Solve the equation $\mathrm{d}y/\mathrm{d}t = e^{-t^2}y^2$ and give the solution in terms of an integral. Describe the behavior of the solution as $t \to \infty$ depending on the initial condition $y(0)$. You may assume that

$$\int_0^\infty e^{-s^2}\,\mathrm{d}s = \frac{\sqrt{\pi}}{2}.$$

Solution

If we separate the variables, we get

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t}e^{-t^2} = y^2 \implies \qquad \frac{1}{y^2}\mathrm{d}y &= e^{-t^2}\mathrm{d}t \\
\int_{y_0}^{y(t)} \frac{1}{y^2}\mathrm{d}y &= \int_0^t e^{-\tilde{t}^2}\,\mathrm{d}\tilde{t} \\
\left[-\frac{1}{y}\right]_{y-y_0}^{t} &= \int_0^t e^{-\tilde{t}^2}\,\mathrm{d}\tilde{t} \\
-\frac{1}{y(t)} + \frac{1}{y_0} &= \int_0^t e^{-\tilde{t}^2}\,\mathrm{d}\tilde{t} \\
y(t) &= \frac{1}{\frac{1}{y_0} - \int_0^t e^{-\tilde{t}^2}\,\mathrm{d}\tilde{t}}.
\end{aligned}$$

Note that the Gaussian integral $\to \sqrt{\pi}/2$ as $t \to \infty$, and it is strictly increasing. Therefore if $1/y_0 = \sqrt{\pi}/2$ then $y(t) \to \infty$ as $t \to \infty$; if $1/y_0 < \sqrt{\pi}/2$, i.e., $y > 2/\sqrt{\pi}$, then $y(t)$ will blow up as $t \to t^*$ for some finite t^* satisfying

$$\frac{1}{y(t^*)} = \int_0^{t^*} e^{-\tilde{t}^2}\,\mathrm{d}\tilde{t}.$$

Lastly, if $1/y_0 > \sqrt{\pi}/2$, i.e., $y_0 < 2/\sqrt{\pi}$, then

$$y(t) \to \frac{1}{\frac{1}{y_0} - \frac{\sqrt{\pi}}{2}} \text{ as } t \to \infty.$$

Problem 8.2

Solve the linear equation

$$\dot{x} + px = q$$

by separation of variables.

Solution

First re-write $\dot{x}$ as $\mathrm{d}x/\mathrm{d}t$:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} + px = q \implies \frac{\mathrm{d}x}{\mathrm{d}t} &= q - px \\ \frac{1}{q-px}\mathrm{d}x &= \mathrm{d}t \\ \int \frac{1}{q-px}\,\mathrm{d}x &= \int \mathrm{d}\tilde{t} \\ -\frac{\ln(q-px)}{p} &= t + c \\ q - px &= Ce^{-pt} \\ \implies x(t) = Ce^{-pt} + \frac{q}{p} & \text{ (of course the } C\text{'s are different but it's just a constant).}\end{aligned}$$

Problem 8.3

Find the general solution of the equation

$$xy' = ky$$

that is valid for $x > 0$.

Solution

First re-write y' as $\mathrm{d}y/\mathrm{d}x$:

$$\begin{aligned}x\frac{\mathrm{d}y}{\mathrm{d}x} = ky \implies \frac{1}{y}\mathrm{d}y &= \frac{k}{x}\mathrm{d}x \\ \int \frac{1}{y}\,\mathrm{d}x &= \int \frac{k}{x}\,\mathrm{d}x \\ \ln(y) &= k\ln(x) + c \\ y &= Cx^k.\end{aligned}$$

Problem 8.6

Previous we have shown, neglecting air resistance, that an apple falling from a height h reaches the ground when $t = \sqrt{2h/g}$. If we include the air resistance the provided that $v \leqslant 0$ the equation becomes

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg + kv^2 \qquad v(0) = 0$$

with $k > 0$. Show that

$$v(t) = -\sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}t\right)$$

and hence that the apple now takes a time

$$t^* = \sqrt{\frac{m}{kg}}\ln\left(e^{kh/m} - \sqrt{e^{2kh/m} - 1}\right)$$

to reach the ground. Check that this coincides with the answer with no air resistance ($t = \sqrt{2h/g}$) as $k \to 0$. Hint: for small x, $e^x \approx 1$ and $\ln(1+x) \approx x$.

Solution

We start by using separation of variables:

$$\begin{aligned}
m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg + kv^2 \implies \frac{\mathrm{d}v}{\mathrm{d}t} &= -g + \frac{k}{m}v^2 \\
\frac{1}{\frac{k}{m}v^2 - g}\mathrm{d}v &= \text{t with } v(0) = 0 \\
\int_0^v \frac{1}{\frac{k}{m}\tilde{v}^2 - g}\,\mathrm{d}\tilde{v} &= \int_0^t \mathrm{d}\tilde{t} \\
\frac{m}{k}\int_0^v \frac{1}{\tilde{v}^2 - \frac{gm}{k}}\,\mathrm{d}\tilde{v} &= t.
\end{aligned}$$

Since

$$\int \frac{1}{a^2 - \tilde{x}^2}\,\mathrm{d}\tilde{x} = \frac{\tanh^{-1}(x/a)}{a},$$

the original equation becomes

$$\begin{aligned}
-\frac{m}{k}\int_0^v \frac{1}{(\sqrt{gm/k})^2 - \tilde{v}^2}\,\mathrm{d}\tilde{v} &= t \\
\left[-\frac{m}{k}\cdot\frac{1}{\sqrt{gm/k}}\tanh^{-1}\left(\frac{\tilde{v}}{\sqrt{gm/k}}\right)\right]_{\tilde{v}=0}^{v} &= t
\end{aligned}$$

$$-\sqrt{\frac{m}{kg}}\tanh^{-1}\left(\frac{v}{\sqrt{gm/k}}\right) = t$$

$$v(t) = -\sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}t\right)$$

and we are done with the first part. For the second part, recall that [well, not really a recall]

$$\int \tanh(kx)\, \mathrm{d}x = \frac{\ln\cosh(kx)}{k},$$

and we want to find t^* satisfying

$$\int_0^{t^*} v(\tilde{t})\, \mathrm{d}\tilde{t} = -h.$$

Substituting the equation for $v(t)$ and dropping negative signs on both sides gives

$$\sqrt{\frac{mg}{k}}\int_0^{t^*}\tanh\left(\sqrt{\frac{gk}{m}}\tilde{t}\right)\mathrm{d}\tilde{t} = h$$

$$\left[\sqrt{\frac{mg}{k}}\sqrt{\frac{m}{gk}}\ln\cosh\left(\sqrt{\frac{gk}{m}}\tilde{t}\right)\right]_{\tilde{t}=0}^{t^*} = h$$

$$\left[\frac{m}{k}\ln\cosh\left(\sqrt{\frac{gk}{m}}\tilde{t}\right)\right]_{\tilde{t}=0}^{t^*} = h$$

$$-\frac{m}{k}\ln\cosh\left(\sqrt{\frac{gk}{m}}t^*\right) = h$$

$$\cosh\left(\sqrt{\frac{gk}{m}}t^*\right) = e^{hk/m}.$$

Again, recall that [no, not a recall!]

$$\cosh x = \frac{e^x + e^{-x}}{2} \implies e^{2x} - 2\cosh x e^x + 1 = e^{2x} - (e^{2x} + e^x e^{-x}) + 1 = 0.$$

Therefore solving the quadratic equation gives

$$e^x = \frac{2\cosh x \pm \sqrt{4\cosh^2 x - 4}}{2} = \cosh x \pm \sqrt{\cosh^2 x - 1} \implies x = \ln(\cosh x \pm \sqrt{\cosh^2 x - 1}).$$

Notice that while $\cosh x \geqslant 1$, it's not hard to verify that $f(x) = \cosh x - \sqrt{\cosh^2 x - 1} < 1$ for all $x > 1$ as this function is strictly decreasing with $f(1) = 1$. Then its natural log < 0. Since t^* should be positive, we discard $\cosh x - \sqrt{\cosh^2 x - 1}$ and claim that

$$\sqrt{\frac{gk}{m}}t^* = \ln(e^{hk/m} - \sqrt{e^{2hk/m} - 1})$$

or

$$t^* = \sqrt{\frac{m}{kg}} \ln(e^{hk/m} - \sqrt{e^{2hk/m} - 1}).$$

Now as $k \to 0$, if we use the hints $e^x \approx 1 + x$ and $\ln(1+x) \approx x$, along with $\sqrt{x} \gg x$ for small x, we get

$$\begin{aligned} t^* &= \sqrt{\frac{m}{kg}} \ln(e^{hk/m} - \sqrt{e^{2hk/m} - 1}) \\ &\approx \sqrt{\frac{m}{kg}} \ln(1 + \frac{hk}{m} - \sqrt{2hk/m}) \\ &\approx \sqrt{\frac{m}{kg}} \ln(1 + \sqrt{2hk/m}) \\ &\approx \sqrt{\frac{m}{kg}} \sqrt{2hk/m} = \sqrt{\frac{2h}{g}}. \end{aligned}$$

Chapter 9 First Order Linear Equations and the Integrating Factor

Problem 9.1

Use an integrating factor to solve the following DEs:

(1)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = x^2,$$

find the general solution and the only solutions that is finite when $x = 0$.

(2)

$$\frac{\mathrm{d}x}{\mathrm{d}t} + tx = 4t$$

and find the solution with $x(0) = 2$.

(3)

$$\frac{\mathrm{d}z}{\mathrm{d}y} = z \tan y + \sin y$$

and find the general solution.

(4)

$$y' + e^{-x} y = 1$$

and find the solution when $y(0) = e$.

(5) With $a > 0$ find the solution of the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \left[a + \frac{1}{t}\right] x = b$$

for a general initial condition $x(1) = x_0$. Also show that $x(t) \to b/a$ as $t \to \infty$. You would get the same result if you replaced $a + 1/t$ by a.

Solution

(1) The integrating factor is $\exp(\int x^{-1}\,\mathrm{d}x) = \exp \ln x = x$. Therefore

$$I(x)y = xy = \int x^3\,\mathrm{d}x = \frac{x^4}{4} + c \implies y = \frac{x^3}{4} + \frac{c}{x}.$$

Since $c/x \to \infty$ as $x \downarrow 0$ and $c/x \to \pm\infty$ as $x \uparrow 0$ for all nonzero c's, the only finite solution for $x = 0$ is if $c = 0$, in which case $y = x^3/4$.

(2) Here the integrating factor is $\exp(\int t\,\mathrm{d}t) = \exp(t^2/2)$. Then

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t}e^{t^2/2} + txe^{t^2/2} = \frac{\mathrm{d}x}{\mathrm{d}t}\left[xe^{t^2/2}\right] &= 4te^{t^2/2} \\
x(t)e^{t^2/2} - x(0) &= \int_0^t 4\tilde{t}e^{\tilde{t}^2/2}\,\mathrm{d}\tilde{t} \\
x(t)e^{t^2/2} - 2 &= \left[4e^{\tilde{t}^2/2}\right]_{\tilde{t}=0}^{t} \\
x(t)e^{t^2/2} &= 4e^{t^2/2} - 2 \\
x(t) &= 4 - 2e^{-t^2/2}.
\end{aligned}$$

(3) First rewrite this DE as

$$\frac{\mathrm{d}z}{\mathrm{d}y} - (\tan y)z = \sin y.$$

Then the integrating factor is $\exp(\int -\tan y\,\mathrm{d}y) = \exp(\ln|\cos y|) = |\cos y|$. Then

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}y}\left[z|\cos y|\right] &= \sin y|\cos y| \\
z(y)|\cos y| &= \int \sin\tilde{y}|\cos\tilde{y}|\,\mathrm{d}\tilde{y} \\
z(y) &= \frac{1}{|\cos y|}\left[-\frac{\cos y|\cos y|}{2} + c\right] = \frac{c}{|\cos y|} - \frac{\cos y}{2}.
\end{aligned}$$

(4) The integrating factor is $\exp(\int e^{-x}\,\mathrm{d}x) = \exp(-e^{-x})$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y\exp(-e^{-x})\right] = \exp(-e^{-x})$$

$$y(x)\exp(-e^{-x}) - y(0)\exp(-e^{-0}) = \int_0^x \exp(-e^{-\tilde{x}})\,\mathrm{d}\tilde{x}$$
$$y(x)\exp(-e^{-x}) - e^{-1+1} = \int_0^x \exp(-e^{-\tilde{x}})\,\mathrm{d}\tilde{x}$$
$$y(x) = \exp(e^{-x}) + \exp(e^{-x})\int_0^x \exp(-e^{-\tilde{x}})\,\mathrm{d}\tilde{x}.$$

(5) The integrating factor is $\exp(\int(a+1/t)\,\mathrm{d}t) = \exp(at + \ln|t|) = |t|e^{at}$. Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[x|t|e^{at}\right] &= b|t|e^{at}\\
x(t)|t|e^{at} - x(1)e^{a} &= \int_1^t b|\tilde{t}|e^{a\tilde{t}}\,\mathrm{d}\tilde{t}\\
x(t)|t|e^{at} - x_0e^{a} &= \left[\frac{b\tilde{t}e^{a\tilde{t}}\operatorname{sgn}(\tilde{t})}{a}\right]_{\tilde{t}=1}^{t} - \int_1^t \frac{be^{a\tilde{t}}\operatorname{sgn}(\tilde{t})}{a}\,\mathrm{d}\tilde{t}\\
&= \frac{bte^{at}\operatorname{sgn}(t) - be^{a}}{a} - \left[\frac{be^{a\tilde{t}}\operatorname{sgn}(t)}{a^2}\right]_{\tilde{t}=1}^{t}\\
&= \frac{bte^{at}\operatorname{sgn}(t) - be^{a}}{a} - \frac{be^{at}\operatorname{sgn}(t) - be^{a}}{a^2}\\
x(t)|t|e^{at} &= \frac{b}{a}\left(te^{at}\operatorname{sgn}(t) - e^{a} - \frac{e^{at}\operatorname{sgn}(t) - e^{a}}{a}\right) + x_0e^{a}\\
x(t) &= \frac{b}{a} + \frac{b}{a}\left(-\frac{e^{a(1-t)}}{|t|} - \frac{1}{at} + \frac{e^{a(1-t)}}{a|t|}\right) + \frac{x_0}{|t|}
\end{aligned}$$

from which we see $x(t) \to b/a$ as $t \to \infty$ which then implies $|t| \to \infty$.

Problem 9.7

Suppose that
$$\frac{\mathrm{d}x}{\mathrm{d}t} \leqslant ax$$
(this is known as a differential *inequality*). Use an appropriate integrating factor and deduce that
$$x(t) \leqslant x(s)e^{a(t-s)}$$
for any t and s.

Solution

First rearrange the term and write it in the standard form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} - ax \leqslant 0.$$

Multiplying both sides by e^{-at} gives

$$\frac{\mathrm{d}x}{\mathrm{d}t}e^{-at} - axe^{-at} = \frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{-at}\right] \leqslant 0.$$

Then integrating both sides from s to t gives

$$\begin{aligned}\left[xe^{-a\tilde{t}}\right]_{\tilde{t}=s}^{t} &\leqslant \int_s^t 0\,\mathrm{d}\tilde{t} \\ x(t)e^{-at} - x(s)e^{-as} &\leqslant 0 \\ x(t)e^{-at} &\leqslant x(s)e^{-as} \\ x(t) &\leqslant x(s)e^{-as+at} = x(s)e^{a(t-s)}.\end{aligned}$$

Problem 9.8

The function $\sin(\omega t)$ and $\cos(\omega t)$ can be written as

$$\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

and

$$\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}.$$

Use these forms to find

$$\int e^{kt}\sin(\omega t)\,\mathrm{d}t,$$

assuming that the usual rules of integration apply to such complex exponentials.

Solution

$$\begin{aligned}\int e^{kt}\sin(\omega t)\,\mathrm{d}t &= \int e^{kt}\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\,\mathrm{d}t \\ &= \int \frac{e^{(k+i\omega)t} - e^{(k-i\omega)t}}{2i}\,\mathrm{d}t \\ &= \frac{1}{2i}\left[\int e^{(k+i\omega)t}\,\mathrm{d}t - \int e^{(k-i\omega)t}\,\mathrm{d}t\right] + C\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2i}\left[\frac{e^{(k+i\omega)t}}{k+i\omega} - \frac{e^{(k-i\omega)t}}{k-i\omega}\right] + C \\
&= \frac{e^{kt}}{2i}\left[\frac{e^{i\omega t}}{k+i\omega} - \frac{e^{-i\omega t}}{k-i\omega}\right] + C \\
&= e^{kt}\left[\frac{(k-i\omega)e^{i\omega t} - (k+i\omega)e^{-i\omega t}}{2i(k+i\omega)(k-i\omega)}\right] + C \\
&= e^{kt}\left[\frac{k(e^{i\omega t} - e^{-i\omega t}) - i\omega(e^{i\omega t} + e^{-i\omega t})}{2i(k^2+\omega^2)}\right] + C \\
&= \frac{e^{kt}[k\sin(\omega t) - \omega\cos(\omega t)]}{k^2+\omega^2} + C.
\end{aligned}
$$

Chapter 10 Two ‘Tricks’ for Nonlinear Equations

Problem 10.1

Check that the following equations are exact and hence solve them.

(1)

$$2x - y\sec^2 x + (x^2 + 2y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

Since

$$\frac{\partial}{\partial y}(2xy - \sec^2 x) = 2x = \frac{\partial}{\partial x}(x^{+}2y)$$

we know this equation is exact. Then it has form $F(x,y) = c$. It follows that

$$F(x,y) = \int 2xy - \sec^2 x \,\mathrm{d}x = x^2 y - \tan x + C(y).$$

Since

$$\frac{\partial}{\partial y}(x^2 y - \tan x + C(y)) = x^2 + C'(y) = x^2 + 2y$$

we know $C(y) = y^2$. Therefore the solution to the original DE is

$$x^2 - \tan x + y^2 = c.$$

(2)

$$(1 + e^x y + xe^x y) + (xe^x + 2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

First check that this DE is exact:

$$\frac{\partial}{\partial y}(1 + e^x y + xe^x y) = e^x + xe^x = \frac{\partial}{\partial x}(xe^x + 2).$$

Now suppose the solution is $F(x, y) = c$, then

$$F(x, y) = \int xe^x + 2 \, \mathrm{d}y = xe^x y + 2y + C(x).$$

Differentiating with respect to x gives

$$e^x y + xe^x y + C'(x) = 1 + e^x y + xe^x y$$

and so $C'(x) = 1 \implies C(x) = x$. Therefore the solution is

$$x + xe^x y + 2y = c.$$

(3)

$$(x \cos y + \cos x)\frac{\mathrm{d}y}{\mathrm{d}x} + \sin y - y \sin x = 0.$$

Solution

First check that this equation is exact:

$$\frac{\partial}{\partial y}(\sin y - y \sin x) = \cos y - \sin x = \frac{\partial}{\partial x}(x \cos y + \cos x).$$

Therefore the solution is of form $F(x, y) = c$ and

$$F(x, y) = \int \sin y - y \sin x \, \mathrm{d}x = x \sin y + y \cos x + C(y).$$

Differentiating with respect to y gives

$$x \cos y + \cos x + C'(y) = x \cos y + \cos x \implies C'(y) = 0.$$

Hence the solution is simply

$$x \sin y + y \cos x = c.$$

(4)

$$e^x \sin y + y + (e^x \cos y + x + e^y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

First verify that this DE is exact:

$$\frac{\partial}{\partial y}(e^x \sin y + y) = e^x \cos y + 1 = \frac{\partial}{\partial x}(e^x \cos y + x + e^y).$$

Then the solution is of form $F(x, y) = c$ and

$$F(x, y) = \int e^x \sin y + y \, \mathrm{d}x = e^x \sin y + xy + C(y).$$

Differentiating with respect to y gives

$$e^x \cos y + x + C'(y) = e^x \cos y + x + e^y$$

which suggests $C(y) = e^y$. Therefore the solution is

$$e^x \sin y + xy + e^y = c.$$

Problem 10.2

Find an integrating factor depending only on x that makes the equation

$$e^{-y} \sec x + 2 \cot x - e^{-y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

exact, and hence find its solution. Hint: $\int \csc x \, \mathrm{d}x = \ln(\csc x - \cot x)$.

Solution

Suppose the integrating factor is $I(x)$, then we have

$$\frac{\partial}{\partial y}\left[I(x)(e^{-y} \sec x)\right] = \frac{\partial}{\partial x}\left[-I(x)e^{-y}\right]$$
$$-I(x)e^{-y} \sec x = -I'(x)e^{-y}$$
$$I(x) \sec x = I'(x).$$

Arranging the last equation as $I'(x) + \sec(x)I(x) = 0$ we see that the solution is

$$I(x) = e^{\int \sec x \, \mathrm{d}x} = \sec x + \tan x.$$

Then the original DE multiplied by $I(x)$ becomes

$$[(\sec x + \tan x)(e^{-y}\sec x + 2\cot x)] + [-(\sec x + \tan x)e^{-y}]\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

of which the solution has form $F(x, y) = C$. Then

$$\int -(\sec x + \tan x)e^{-y} \, \mathrm{d}y = (\sec x + \tan x)e^{-y} + C(x)$$

and

$$\begin{aligned}\frac{\partial}{\partial x}\left[(\sec x + \tan x)e^{-y} + C(x)\right] &= \sec x(\sec x + \tan x)e^{-y} + C'(x) \\ &= \sec x(\sec x + \tan x)e^{-y} + 2\cot x(\sec x + \tan x).\end{aligned}$$

Hence

$$\begin{aligned}C(x) &= \int 2\cot x(\sec x + \tan x) \, \mathrm{d}x \\ &= \int 2\csc x + 2 \, \mathrm{d}x \\ &= 2\ln(\csc x - \cot x) + 2 + c,\end{aligned}$$

and so the solution of the DE is given by

$$(\sec x + \tan x)e^{-y} + 2\ln(\csc x - \cot x) + 2 = C.$$

Problem 10.3

Show that any equation that can be written in the form

$$f(x) + g(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is exact, and find its solution in terms of integrals of f and g.

Solution

If $f(x)$ is a function in terms of only x and g a function in terms of only y then

$$\frac{\partial f}{\partial y} = 0 = \frac{\partial g}{\partial x}$$

which makes the DE exact. Let the solution be of form $F(x,y) = c$ and we have

$$F(x,y) = \int f(x)\, \mathrm{d}x + C(y).$$

Since

$$\frac{\partial}{\partial y}\left[\int f(x)\, \mathrm{d}x + C(y)\right] = C'(y) = g(y)$$

we know $C(y) = \int g(y)\, \mathrm{d}y$. Hence the solution is

$$\int f(x)\, \mathrm{d}x + \int g(y)\, \mathrm{d}y = c.$$

Now find the solution of

(1)

$$V'(x) + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

Let the solution be $F(x,y) = c$. Integrating $V'(x)$ with respect to x gives

$$F(x,y) = \int V'(x)\, \mathrm{d}x = V(x) + C(y)$$

whereas

$$\frac{\partial}{\partial y}[V(x) + C(y)] = 2y.$$

Therefore $C(y) = y^2$ and the solution is

$$V(x) + y^2 = c. \tag{1}$$

(2) and of

$$\left(\frac{1}{y} - a\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x} - b = 0$$

Solution

Applying the result derived from (1), the solution is

$$\int \frac{1}{y} - a \,\mathrm{d}y + \int \frac{2}{x} - b \,\mathrm{d}x = c,$$

and arranging terms gives

$$\ln y - ay + 2\ln x - bx = c.$$

Problem 10.4

By substituting $u = y/x$, solve the following homogeneous equations:

(1)

$$xy + y^2 + x^2 - x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

If we let $u = y/x$ then $y = xu$ and $\mathrm{d}y/\mathrm{d}x = u + x\mathrm{d}u/\mathrm{d}x$. Diving the entire DE by x^2 we get

$$\frac{y}{x} + \frac{y^2}{x^2} + 1 - \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \frac{y^2}{x^2} + 1$$

which can be re-written as

$$u + x\frac{\mathrm{d}u}{\mathrm{d}x} = u + u^2 + 1 \implies x\frac{\mathrm{d}u}{\mathrm{d}x} = u^2 + 1 \implies \frac{1}{u^2+1}\mathrm{d}u = \frac{1}{x}\mathrm{d}x.$$

Integrating both sides gives

$$\int \frac{1}{u^2+1}\,\mathrm{d}u = \int \frac{1}{x}\,\mathrm{d}x \implies \arctan(u) = \arctan(y/x) = \ln x + c.$$

Thus the solution is

$$y = x\tan(\ln x + c).$$

(2)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x^2 + t\sqrt{t^2 + x^2}}{tx}.$$

Solution

Again, letting $u = x/t$ we have $\mathrm{d}x/\mathrm{d}t = u + t\mathrm{d}u/\mathrm{d}t$. Now the original equation becomes

$$u + t\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{x^2 + t\sqrt{t^2 + x^2}}{tx} = \frac{x}{t} + \sqrt{t^2/x^2 + 1} = u + \sqrt{1/u^2 + 1}.$$

Rearranging this equation gives

$$t\frac{\mathrm{d}u}{\mathrm{d}t} = \sqrt{1/u^2 + 1} \implies \frac{1}{\sqrt{1/u^2 + 1}}\mathrm{d}u = \frac{1}{t}\mathrm{d}t,$$

so integrating both sides and factoring out the $1/u^2$ in the LHS gives

$$\int \frac{u}{\sqrt{1 + u^2}}\,\mathrm{d}u = \int \frac{1}{t}\,\mathrm{d}t \implies \sqrt{1 + u^2} = \ln t + c.$$

Therefore

$$u = \pm\sqrt{(\ln t + c)^2 - 1}$$

and

$$x(t) = \pm t\sqrt{(\ln t + c)^2 - 1}.$$

Problem 10.5

Solve

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kx - x^2$$

by substituting $u = x^{-1}$ and show u satisfies the linear equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 1 - ku.$$

Solve this equation for $u(t)$ and hence find the solution $x(t)$.

Solution

If we let $u = x^{-1}$ then

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -x^{-2}\frac{\mathrm{d}x}{\mathrm{d}t}$$

and hence

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -x^{-2}(kx - x^2)$$

$$\implies \frac{\mathrm{d}u}{\mathrm{d}t} = -x^{-1}k + 1$$
$$\implies \frac{\mathrm{d}u}{\mathrm{d}t} = 1 - ku$$

as claimed by the question. Now we try to solve this, i.e., $\frac{\mathrm{d}u}{\mathrm{d}t} + ku = 1$. Clearly the integrating factor is e^{kt}:

$$\begin{aligned} e^{kt}\frac{\mathrm{d}u}{\mathrm{d}t} + e^{kt}ku &= e^{kt} \\ \frac{\partial}{\partial t}\left[e^{kt}u\right] &= e^{kt} \\ \int \frac{\partial}{\partial t}\left[e^{k\tilde{t}}u\right]\,\mathrm{d}\tilde{t} &= \int e^{k\tilde{t}}\,\mathrm{d}\tilde{t} \\ e^{kt}u &= \frac{1}{k}e^{kt} + c \\ u(t) &= \frac{1}{k} + ce^{-kt}. \end{aligned}$$

Substituting $x = u^{-1}$ into this equation we get the final solution:

$$x(t) = \frac{1}{1/k + ce^{-kt}}.$$

Problem 10.6

Use an appropriate substitution to solve the equation

$$\dot{x} = x(\kappa^2 - x^2).$$

Solution

Notice that this equation can be re-written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \kappa^2 x - x^3,$$

a *Bernoulli equation.* If we let $u := x^{-2}$ then

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= -2x^{-3}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -2x^{-3}(\kappa^2 x - x^3) \\ &= 2 - 2\kappa^2 x^{-2} \\ &= 2 - 2\kappa^2 u. \end{aligned}$$

Then, for this linear equation, we can let $I(x) = \exp(\int 2\kappa^2\,\mathrm{d}t) = e^{2\kappa^2 t}$ be the integrating factor, and the equation becomes

$$\begin{aligned}
e^{2\kappa^2 t}\frac{\mathrm{d}u}{\mathrm{d}t} + e^{2\kappa^2 t}2\kappa^2 u &= 2e^{2\kappa^2 t} \\
\frac{\partial}{\partial t}\left[e^{2\kappa^2 t}u\right] &= 2e^{2\kappa^2 t} \\
\int \frac{\partial}{\partial t}\left[e^{2\kappa^2 t}u\right]\,\mathrm{d}t &= \int 2e^{2\kappa^2 t}\,\mathrm{d}t \\
e^{2\kappa^2 t}u &= \frac{e^{2\kappa^2 t}}{\kappa^2} + C \\
u(t) &= \frac{1}{\kappa^2} + Ce^{-2\kappa^2 t}.
\end{aligned}$$

Substituting $u = x^{-2} \implies x = u^{-1/2}$ back to this equation we get

$$x(t) = \sqrt{\frac{\kappa^2}{1 + C\kappa^2 e^{-2\kappa^2 t}}}.$$