11 Second Order Linear Equations: General Theory

Problem 11.2

Show that the Wronskian of two solutions $x_1(t)$ and $x_2(t)$ of the second order differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + p_1(t)\frac{\mathrm{d}x}{\mathrm{d}t} + p_2(t)x = 0 \tag{1}$$

satisfies

$$\dot{W}(t) = -p_1(t)W(t).$$

[Hint: write $W(t) = x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t)$, differentiate, and use the fact that $x_1(t)$ and $x_2(t)$ satisfy the equation above.] Deduce either $W(t) = 0$ for all t or $W(t) \neq 0$ for all t.

Solution

Differentiating $W(t) = x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t)$ with respect to t gives

$$\begin{aligned}
\dot{W}(t) &= x_1\frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} + \frac{\mathrm{d}x_1}{\mathrm{d}t}\frac{\mathrm{d}x_2}{\mathrm{d}t} - x_2\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} - \frac{\mathrm{d}x_1}{\mathrm{d}t}\frac{\mathrm{d}x_2}{\mathrm{d}t} \\
&= x_1[-p_1(t)\frac{\mathrm{d}x_2}{\mathrm{d}t} - p_2(t)x_2] - x_2[-p_1(t)\frac{\mathrm{d}x_1}{\mathrm{d}t} - p_2(t)x_1] \\
&= -p_1(t)[x_1\frac{\mathrm{d}x_2}{\mathrm{d}t} + x_2\frac{\mathrm{d}x_1}{\mathrm{d}t}] \\
&= -p_1(t)W(t).
\end{aligned}$$

Here we have acquired the differential equation

$$\dot{W}(t) + p_1(t)W(t) = 0.$$

The integrating factor is $I(t) = \exp(\int p_1(t)\,\mathrm{d}t)$ and so

$$\begin{aligned}
I(t)\dot{W}(t) + I(t)p_1(t)W(t) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}[I(t)W(t)] &= 0 \\
W(t) &= Ce^{-\int p_1(t)\,\mathrm{d}t}.
\end{aligned}$$

Since $e^{I(t)}$ is never zero, either $C = 0 \implies W(t) = 0$ for all t or $C \neq 0 \implies W(t) \neq 0$ for all t.

Problem 11.3

We have seen that if x_1 and 2 are two solutions of a linear differential equation, then they are linearly independent if and only if their Wronskian is nonzero. The simple example of this question shows that this is not true for general functions that are not the solutions of some differential equation.

(1) Check carefully that if $(t) = t^2|t|$ then $\mathrm{d}f/\mathrm{d}t = 3t|t|$. This is easy when $t \neq 0$. You will have to use the formal definition of derivative at $t = 0$.

(2) Let

$$f_1(t) = t^2|t| \text{ and } f_2(t) = t^3.$$

Show that although these two functions are linearly independent on $\mathbb{R}$, their Wronskian is identically zero.

Solution

When $t > 0$, $t^2|t| = t^3$ and $\mathrm{d}f/\mathrm{d}t = 3t^2 = 3t|t|$. Likewise, when < 0, $t^2|t| = -t^3$ and $\mathrm{d}f/\mathrm{d}t = -3t^2 = 3t|t|$. When $t = 0$,

$$\dot{f}(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2|h|}{h} = \lim_{h\to 0} h|h| = 0.$$

Therefore $\mathrm{d}f/\mathrm{d}t = 3t|t|$.

For the second part,

$$W[f_1 f_2](t) = \begin{vmatrix} t^2|t| & t^3 \\ 3t|t| & 3t^2 \end{vmatrix} = 3t^4|t| - 3t^4|t| = 0.$$

(They are clearly independent.)

12 Homogeneous 2^{nd}-Order DEs with Constant Coeff.

Problem 12.2

If the roots of the auxiliary equation are $k_1 > 0$ and $-k_2 < 0$ then the solution is

$$x(t) = Ae^{k_1 t} + Be^{-k_2 t}.$$

For most choices of initial conditions

$$x(0) = x_0 \text{ and } \dot{x}(0) = y_0$$

we will have $x(t) \to \pm\infty$ as $t \to \infty$. However, there are some special initial conditions for which $x(t) \to 0$ as $t \to \infty$. Find the relationship between x_0 and y_0 that ensures this.

Solution

For any nonzero A, the term $Ae^{k_1 t}$ grows exponentially, and $Be^{-k_2 t}$ has no way to offset this. Therefore if $x(t) \to 0$ then $A = 0$. Now we solve the initial condition problem

$$x(t) = Be^{-k_2 t} \text{ with } x(0) = x_0 \text{ and } \dot{x}(0) = y_0.$$

This gives

$$\begin{cases} B = x_0 \\ -k_2 B = y_0 \end{cases} \implies k_2 x_0 + y_0 = 0 \text{ being a necessary condition to prevent } x(t) \text{ blowing up.}$$

Problem 12.3

Solutions of linear equations with constant coefficients cannot blow up in finite time; it follows that their solutions exist for all $t \in \mathbb{R}$. To see this, we will consider

$$\ddot{x} + p\dot{x} + qx = 0 \text{ with } x(0) = x_0 \text{ and } \dot{x}(0) = y_0$$

for $t \geqslant 0$ (a similar argument applies for $t \leqslant 0$). By setting $y = \dot{x}$, we can rewrite this as a coupled pair of first order equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -py - qx. \end{aligned}$$

Show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) = (1 - q)xy - py^2$$

and hence that

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) \leqslant (1 + |q| + 2|p|)(x^2 + y^2).$$

Use the result of Exercise 9.7 to deduce that, for $t \geqslant 0$,

$$x(t)^2 + y(t)^2 \leqslant (x(0)^2 + y(0)^2)e^{(1+|q|+2|p|t)},$$

showing that finite-time blowup is impossible. [Hint: $xy \leqslant (x^2 + y^2)/2$.]

(The same argument works, essentially unchanged, for

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0$$

provided that $|p(t) \leqslant p\,|$ and $|q(t)| \leqslant q$ for all $t \in \mathbb{R}$.)

Solution

Straightforward computation...

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) &= \frac{1}{2}(2x\dot{x} + 2y\dot{y}) \\
&= x\dot{x} + y\dot{y} \\
&= xy + y(-py - qx) \\
&= xy(1 - q) - py^2 && \text{(first step done)} \\
&\leqslant xy(1 + |q|) + |p|y^2 \\
&\leqslant \frac{1}{2}(x^2 + y^2)(1 + |q|) + |p|y^2 && \text{(Cauchy)} \\
&\leqslant \frac{1}{2}\left[(x^2 + y^2)(1 + |q|)\right] + |p|x^2 + |p|y^2 \\
&= \frac{1}{2}\left(1 + |q| + 2|p|\right)(x^2 + y^2). && \text{(second step done)}
\end{aligned}$$

Exercise 9.7 states that

$$\frac{\mathrm{d}x}{\mathrm{d}t} \leqslant ax \implies x(t) \leqslant x(s)e^{a(t-s)} \text{ for any } t, s.$$

Letting $a := 1 + |q| + 2|p|$ and $t := 0$, we have

$$x(t)^2 + y(t)^2 \leqslant (x(0)^2 + y(0)^2)e^{(1+|q|+2|p|)t} \text{ as desired.}$$

13 Oscillations

Problem 13.2

The acceleration due to gravity in fact depends on the distance R from the center of the Earth: $g = GM/R^2$, where M is the mass of the Earth and G Newton's gravitational constant. Show that the period of oscillation of a pendulum will increase as it is taken higher.

Solution

The ideal pendulum oscillates with period $2\pi\sqrt{L/g} \propto R$. One-liner isn't it?

Problem 13.4

The buoyancy force on an object is equal to the weight of water that it displaces. If an object has mass M and displaces a volume V of water then the forces on it are $Mg - Vg$, in units for which the density of water is 1.

A bird of mass m is sitting on a cylindrical buoy of density ρ, radius R, and height h, which is floating at rest. How much of the buoy lies below the surface?

The bird flies away. Show that the buoy now bobs up and down, with the amount below the surface oscillating about ρh with period $2\pi\sqrt{\rho h/g}$ and amplitude $m/\pi R^2$.

Solution

Initially, without the bird, the buoy is floating with height ρh under water: check that, while with ρh under water, the buoyancy force is $1 \cdot g \cdot \pi R^2 \rho h$ whereas the buoy's weight is $\pi R^2 h \cdot \rho \cdot g$. They are indeed equal and the buoy is stationary. To simplify notations we will now denote the mass of buoy as M. Thus $M := \pi R^2 \rho h$.

When the bird comes, the total weight of the buoy-bird combo is $M + m$. Let $x(t)$ be the function that describes the depth to which the buoy is immersed at time t. It follows that

$$x(0)\pi R^2 g = (M + m)g \implies x(0) = \frac{M + m}{\pi R^2}.$$

After the bird left, the motion of buoy can be described by the differential equation

$$\sum F = M\ddot{x} = Mg - \pi R^2 gx = \pi R^2 g(\rho h - x).$$

The equilibrium is achieved when $\ddot{x} = 0$. Therefore setting $\ddot{x} = 0$ gives $x = \rho h$, i.e., the buoy oscillates about depth of ρh. Now if we define $y := x - \rho h$ and "shift" the center of oscillation to the origin, we have

$$M\ddot{x} = M\ddot{y} = -\pi R^2 gy \implies M\ddot{y} + \pi R^2 gy = 0,$$

a homogeneous equation. Setting the solution as $y = ce^{kt}$ we get the auxiliary equation

$$Mk^2 + \pi R^2 g = 0 \implies k^2 = -\frac{\pi R^2 g}{M} = -\frac{g}{\rho h}.$$

Therefore $k = \pm i\sqrt{g/\rho h} := i\omega$ and the solution is of form

$$y(t) = A\cos\omega t + B\sin\omega t.$$

Now we need to plug in the initial conditions. At $t = 0$ we know $y(0) = x(0) - \rho h = m/\pi R^2$. We also know that the buoy starts from stationary so $y'(0) = 0$. Hence

$$\begin{cases} A = m/\pi R^2 \\ \omega B\cos = 0 \implies B = 0 \end{cases}$$

and $y(t) = m\cos\omega t/(\pi R^2)$. We conclude that

$$x(t) = \frac{m}{\pi R^2}\cos\left(\sqrt{\frac{g}{\rho h}}t\right) + \rho h.$$

It all becomes clear now that $x(t)$ indeed oscillates about ρh with period $2\pi\sqrt{\rho h/g}$ and amplitude $m/\pi R^2$.

Problem 13.6

A right circular cone, of height h, density ρ, and with base radius R, is placed point downward in a lake. Assuming the apex remains point vertically downwards, show that if the cone is submerged to depth x then

$$\ddot{x} = g - \left(\frac{x}{h}\right)^3 \frac{g}{\rho}.$$

(You need not solve this equation.) At equilibrium how far is the cone submerged?

Solution

On one hand the weight of the cone is given by $\frac{1}{3}\pi R^2 h\rho g$. On the other hand, the buoyancy force is

$\frac{1}{3}\pi\left(\frac{xR}{h}\right)^2 xg$. Therefore

$$\sum F = \frac{1}{3}\pi R^2 h\rho\ddot{x} = \frac{1}{3}\pi R^2 h\rho g - \frac{1}{3}\pi\left(\frac{xR}{h}\right)^2 xg$$

which simplifies to

$$h\rho\ddot{x} = h\rho g - \frac{x^3}{h^2}g \implies \ddot{x} = g - \left(\frac{x}{h}\right)^3 \frac{g}{\rho}.$$

At equilibrium we would have $\ddot{x} = 0$, so

$$g = \left(\frac{x}{h}\right)^3 \frac{g}{\rho} \implies x = h\sqrt[3]{\rho}.$$

Problem 13.7

A dashpot is a device designed to add damping to a system, consisting essentially of a plunger in a cylinder of liquid or gas. It produces a resisting force proportional to the velocity, precisely the kind of damping that we used in our model

$$m\ddot{x} + \mu\dot{x} + kx = 0, \tag{2}$$

with μ indicating the "strength" of the dashpot. Dashpots are used in a variety of applications, for example, cushioning the opening mechanism on a tape recorder, or in car shock absorbers. A mass-spring-dashpot system consists of a mass attached to a spring and a dash- pot, and is shown in the figure. A weight of mass 10 kg is attached to a spring with spring constant 5, and to a dashpot of strength μ. How strong should the dashpot be to ensure that the system is over-damped? What would the period of oscillations be if $\mu = 14$?

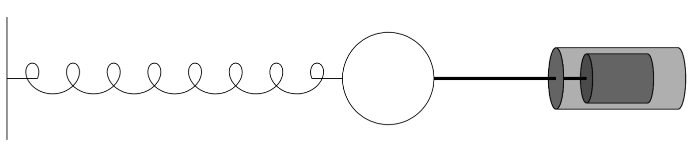


Fig. 13.9. A mass-spring-dashpot system.

Solution

Recall that the auxiliary equation needs to have no real root for the system to be over-damped, in this case $\mu \geqslant \sqrt{4 \cdot 5 \cdot 10} = 2\sqrt{50}$.

For the second part, if $m = 10, \mu = 14$, and $k = 5$, assuming the solution is of form $x(t) = ce^{kt}$, we have

$$10k^2 + 14k + 5 = 0 \implies k_{1,2} = \frac{-7 \pm i}{10}.$$

Therefore the general solution is

$$x(t) = e^{7t/10}\left[A\cos(0.1t) + B\sin(0.1t)\right]$$

which gives a period of 20π.

Problem 13.9

In the case of critical damping (see Section 13.3), the general solution of

$$\ddot{x} + \lambda\dot{x} + \omega^2 x = 0$$

is of the form

$$x(t) = (A + Bt)\mathrm{e}^{-\lambda t/2}$$

Show that if $\lambda A < 2B$ then $x(t)$ increases initially, reaching its maximum value at

$$t = \frac{2}{\lambda} - \frac{A}{B}.$$

Solution

More computation...

$$\begin{aligned}\dot{x}(t) &= -\frac{\lambda}{2}Ae^{-\lambda t/2} + Be^{-\lambda t/2} - \frac{\lambda}{2}tBe^{-\lambda t/2}\\ &= e^{-\lambda t/2}\left(B - \frac{\lambda}{2}(A + Bt)\right)\\ &= \frac{1}{2}e^{-\lambda t/2}(2B - \lambda A - \lambda Bt).\end{aligned}$$

Therefore if $\lambda A < 2B$ we have

$$\dot{x}(0) = \frac{1}{2}e^{-1}(\underbrace{2B - \lambda A}_{\text{positive}} + 0) > 0.$$

Since $\dot{x}(t)$ is strictly decreasing, when $2B - \lambda A = \lambda Bt$, i.e., when $t = \dfrac{2}{\lambda} - \dfrac{A}{B}$ (assuming $B, \lambda \neq 0$), $x(t)$ reaches its maximum.

14 Inhomogeneous Second Order Linear Equations

Problem 14.1.3

Find the general solution to

$$\ddot{x} + \dot{x} - 2x = 3e^{-t}$$

and also find the solution with $x(0) = 3$ and $\dot{x}(0) = 0$.

Solution

We first need to find the complementary function, i.e., the solutions of

$$\ddot{x} + \dot{x} - 2x = 0.$$

"Guessing" x would be of form e^{kt} gives

$$k^2e^{kt} + ke^{kt} - 2e^{kt} = (k^2 + k - 2)e^{kt} = 0 \implies k^2 + k - 2 = (k-1)(k+2) = 0,$$

and so $x_h(t) = Ae^t + Be^{-2t}$. Now for the particular integral we "guess" it's of form Ce^{-t}. Then

$$\begin{cases} \ddot{x}_p = Ce^{-t} \\ \dot{x}_p = -Ce^{-t} \end{cases} \implies \ddot{x}_p + \dot{x}_p - 2x_p = -2Ce^{-t} = 3e^{-t}.$$

Therefore $C = -3/2$, the particular solution is $x_p = -3e^{-t}/2$, and the general solution is

$$x(t) = Ae^t + Be^{-2t} - \frac{3}{2}e^{-t}.$$

Now plug in the initial conditions:

$$\begin{cases} x(0) = 3 \implies A + B = 4.5 \\ \dot{x}(0) = 0 \implies A - 2B = -1.5 \end{cases} \implies \begin{cases} A = 2.5 \\ B = 2 \end{cases} \implies x(t) = \frac{5}{2}e^t + 2e^{-2t} - \frac{3}{2}e^t.$$

Problem 14.1.4

Find the general solution to

$$\ddot{x} + \omega^2 x = \sin(\alpha t) \text{ where } \alpha \neq \omega.$$

Also find the solution with $x(0) = 6$ and $\dot{x}(0) = 0$.

Solution

Again, for the complementary function, we "guess" it's of form e^{kt}. Then

$$k^2e^{kt} + \omega^2 e^{kt} = (k^2 + \omega^2)e^{kt} = 0 \implies k^2 + \omega^2 = 0 \implies k = \pm\omega i.$$

Therefore $x_h(t) = A\cos(\omega t) + B\sin(\omega t)$. Now we "guess" the particular integral is of form $C\sin(\alpha t) + D\cos(\alpha t)$. Then its second derivative is $-\alpha^2(C\sin(\alpha t) + D\cos(\alpha t))$. Therefore

$$(\omega^2 - \alpha^2)(C\sin(\alpha t) + D\cos(\alpha t)) = \sin(\alpha t) \implies \begin{cases} C = \dfrac{1}{\omega^2 - \alpha^2} \\ D = 0 \end{cases}$$

and thus the general solution is

$$x(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{\sin(\alpha t)}{\omega^2 - \alpha^2}.$$

Now plug in the initial conditions:

$$\begin{cases} x(0) = A = 6 \\ \dot{x}(0) = \omega B + \dfrac{\alpha}{\omega^2 - \alpha^2} = 0 \implies B = -\dfrac{\alpha}{\omega(\omega^2 - \alpha^2)} \end{cases}$$

and so the solution to this initial condition problem is

$$x(t) = 6\cos(\omega t) + \frac{\omega\sin(\alpha t) - \alpha\sin(\omega t)}{\omega(\omega^2 - \alpha^2)}.$$

Problem 14.1.7

Now find the general solution to

$$\ddot{x} + \omega^2 x = \sin(\alpha t) \text{ but with } \alpha = \omega.$$

Solution

The complementary function is still $x_h(t) = A\cos(\omega t) + B\sin(\omega t)$. Now that $\sin(\omega t) = \sin(\alpha t)$, our "guess" for particular integral becomes $Ct\sin(\alpha t) + Dt\cos(\alpha t)$. Then

$$\dot{x}_p = Ct\alpha\cos(\alpha t) + C\sin(\alpha t) - Dt\alpha\sin(\alpha t) + D\cos(\alpha t)$$

and

$$\begin{aligned}\ddot{x}_p &= -Ct\alpha^2\sin(\alpha t) + C\alpha\cos(\alpha t) + C\alpha\cos(\alpha t) - Dt\alpha^2\cos(\alpha t) - D\alpha\sin(\alpha t) - D\alpha\sin(\alpha t)\\ &= 2C\alpha\cos(\alpha t) - 2D\alpha\sin(\alpha t) - Ct\alpha^2\sin(\alpha t) - Dt\alpha^2\cos(\alpha t).\end{aligned}$$

Therefore

$$\ddot{x}_p + \omega^2 x_p = 2C\alpha\cos(\alpha t) - 2D\alpha\sin(\alpha t) = \sin(\alpha t)$$

which gives $C = 0$ and $D = -1/2\alpha$, so the general solution is

$$x(p) = A\cos(\omega t) + B\sin(\omega t) - \frac{t\cos(\omega t)}{2\omega}.$$

15 Resonance

Problem 15.1

For $\alpha \neq \omega$ show that the solution of the equation

$$\ddot{x} + \omega^2 x = \cos(\alpha t)$$

with $x(0) = \dot{x}(0) = 0$ is

$$x(t) = \frac{1}{\omega^2 - \alpha^2}(\cos(\alpha t) - \cos(\omega t)).$$

Solution

First, the complementary function to $\ddot{x} + \omega^2 x = 0$ is provided by $x_h(t) = A\cos(\omega t) + B\sin(\omega t)$. Now, for the particular integral, we "guess" that it is of form $x_p(t) = C\cos(\alpha t)$ [no need to include a term $D\sin(\alpha t)$ because $\dot{x}$ does not appear in the equation]. Then

$$\begin{cases} \dot{x}_p = -C\alpha\sin(\alpha t) \\ \ddot{x}_p = -C\alpha^2\cos(\alpha t) \end{cases} \implies \ddot{x}_p + \omega^2 x_p = C(\omega^2 - \alpha^2)\cos(\alpha t).$$

It follows that $C = 1/(\omega^2 - \alpha^2)$ and so the general solution is

$$x(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{\cos(\alpha t)}{\omega^2 - \alpha^2}.$$

Plugging in $x(0) = 0$ gives $A + 1/(\omega^2 - \alpha^2) = 0 \implies A = -1/(\omega^2 - \alpha^2)$. Plugging in $\dot{x}(0) = 0$ gives

$$\left[-A\omega\sin(\omega t) + B\omega\cos(\omega t) + \frac{\alpha\sin(\alpha t)}{\alpha^2 - \omega^2}\right]_{t=0} = B\omega = 0 \implies B = 0.$$

Therefore the solution to the given initial conditions is

$$x(t) = \frac{1}{\omega^2 - \alpha^2}(\cos(\alpha t) - \cos(\omega t)), \text{ as stated.}$$

Problem 15.2

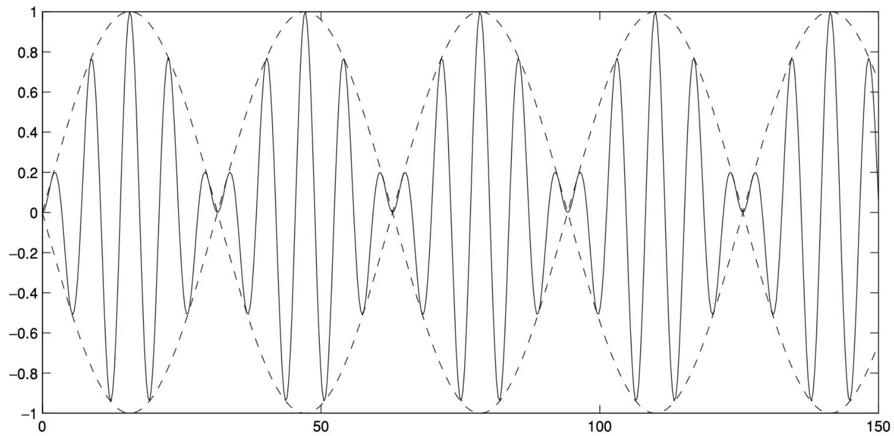
Use the double angle formulae

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

to find an expression for $\cos x - \cos y$ as a product of two sine functions, and hence the solution to the previous problem becomes

$$\frac{2}{\omega^2 - \alpha^2}\sin\left(\frac{(\omega+\alpha)t}{2}\right)\sin\left(\frac{(\omega-\alpha)t}{2}\right).$$

If α is close to ω then $|\omega + \alpha| \gg |\omega - \alpha|$; one of the two terms oscillates much faster than the other. A graph of such an expression when $\omega = 1$ and $\alpha = 0.8$ is shown below. The periodic variation of the amplitude of the basic oscillation is known as **beating**. You can hear this when, for example, two flutes play slightly out of tune with each other.



Solution

Simple:

$$\begin{cases} \cos(x+y) = \cos x \cos y - \sin x \sin y \\ \cos(x-y) = \cos x \cos y + \sin x \sin y \end{cases} \implies \cos(x-y) - \cos(x+y) = 2 \sin x \sin y.$$

Letting $\alpha t = x - y$ and $\omega t = x + y$ gives

$$\begin{cases} x = (\omega + \alpha)t/2 \\ y = (\omega - \alpha)t/2 \end{cases} \implies x(t) = \frac{2}{\omega^2 - \alpha^2} \sin\left(\frac{(\omega+\alpha)t}{2}\right) \sin\left(\frac{(\omega-\alpha)t}{2}\right).$$

Problem 15.3

When $\alpha = \omega$ show that the solution of 15.1 with $x(0) = \dot{x}(0) = 0$ is

$$x(t) = \frac{t \sin(\omega t)}{2\omega}.$$

Also recover this solution from by letting $a \to \omega$ $(\alpha \neq \omega)$ in 15.1's solution and using L'Hôpital's rule.

Solution

Even simpler:

$$\begin{aligned}\lim_{\alpha\to\omega}\frac{1}{\omega^2-\alpha^2}(\cos(\alpha t)-\cos(\omega t)) &= \lim_{\alpha\to\omega}\frac{\frac{\partial}{\partial\alpha}\left[\cos(\alpha t)-\cos(\omega t)\right]}{\frac{\partial}{\partial\alpha}\left[\omega^2-\alpha^2\right]}\\ &= \lim_{\alpha\to\omega}\frac{-t\sin(\alpha t)}{-2\alpha}\\ &= \frac{t\sin(\omega t)}{2\omega}.\end{aligned}$$

[Why?]

Problem 15.4

A model for the vibrations of a wine glass is

$$\ddot{x}+\lambda\dot{x}+\omega^2 x=0,$$

where λ and ω are constants. Suppose that when struck the glass vibrates at 660 Hz. Show that

$$\sqrt{4\omega^2-\lambda^2}=2640\pi.$$

If it takes about 3 seconds for the sound to die away, and this happens when the original vibrations have reduced to 1/100 of their initial level, show that

$$\lambda=\frac{2\log 100}{3},$$

and hence that $\lambda = 3.07$ and $\omega = 4.15\cdot 10^3$ (to 3 significant figures).

The glass can stand deforming only to $x\approx 1$. A pure tone at 660 Hz is produced at D decibels and aimed at the glass, forcing it at its natural frequency, so that the vibrations are now modelled by

$$\ddot{x}+\lambda\dot{x}+\omega^2 x=\frac{10^{0.1D-8}}{3}\cos(1320\pi)t.$$

How loud should the sound be in order to shatter the glass? (The strange factor in front of the forcing produces roughly the correct volume level.)

Solution

"Guessing" that $x(t)$ is of form e^{kt} gives

$$k^2 + \lambda k + \omega^2 = 0 \implies k = -\frac{\lambda}{2} \pm \frac{\sqrt{\lambda^2 - 4\omega^2}}{2}.$$

Since we are given that the glass vibrates, this must be an under-damping and the second term must be imaginary. Then

$$k = -\frac{k}{2} \pm i\frac{\sqrt{4\omega^2 - \lambda^2}}{2} \implies x(t) = e^{-\lambda t/2}\left[A\cos\left(t\frac{\sqrt{4\omega^2 - \lambda^2}}{2}\right) + B\sin\left(t\frac{\sqrt{4\omega^2 - \lambda^2}}{2}\right)\right].$$

It follows that the frequency is given by $(\sqrt{4\omega^2 - \lambda^2}/2)/2\pi = 660$ Hz, and so $\sqrt{4\omega^2 - \lambda^2} = 2640\pi$, as desired.

To reduce the magnitude to 1/100 in 3 seconds, we want

$$e^{-3\lambda/2} = \frac{1}{100} \implies -\frac{3\lambda}{2} = \ln\left(\frac{1}{100}\right) \implies \lambda = \frac{2\ln 100}{3},$$

and plugging this into $\sqrt{4\omega^2 - \lambda^2} = 2640\pi$ gives $\omega = 4.15 \cdot 10^3$.

The text provides the derivation of finding the particular integral of

$$\ddot{x} + \lambda\dot{x} + \omega^2 x = a\cos(\alpha t)$$

and its magnitude, but nevertheless I'd like to derive it on my own one more time. "Guessing" that the particular integral is of form $x_p(t) = C\sin(\alpha t) + D\cos(\alpha t)$ we get

$$\begin{cases} \dot{x}_p = C\alpha\cos(\alpha t) - D\alpha\sin(\alpha t) \\ \ddot{x}_p = -C\alpha^2\sin(\alpha t) - D\alpha^2\cos(\alpha t) \end{cases}$$

and

$$\ddot{x} + \lambda\dot{x} + \omega^2 x = a\cos(\alpha t) = [-C\alpha^2 - \lambda D\alpha + C\omega^2]\sin(\alpha t) + [-D\alpha^2 + \lambda C\alpha + D\omega^2]\cos(\alpha t).$$

Solving the equation gives

$$\begin{cases} C(\omega^2 - \alpha^2) = \lambda D\alpha \\ D(\omega^2 - \alpha^2) = a - \lambda C\alpha \end{cases} \implies \left[\frac{C(\omega^2 - \alpha^2)}{\lambda\alpha}\right](\omega^2 - \alpha^2) = a - \lambda C\alpha \implies C = \frac{a\lambda\alpha}{(\omega^2 - \alpha^2)^2 + (\lambda\alpha)^2}.$$

Thus

$$C = \frac{a\lambda\alpha}{(\omega^2 - \alpha^2)^2 + (\lambda\alpha)^2} \text{ and } D = \frac{a(\omega^2 - \alpha^2)}{(\omega^2 - \alpha^2)^2 + (\lambda\alpha)^2}$$

and the particular integral is given by

$$x_p(t) = \frac{a\lambda\alpha}{(\omega^2 - \alpha^2)^2 + (\lambda\alpha)^2}\sin(\alpha t) + \frac{a(\omega^2 - \alpha^2)}{(\omega^2 - \alpha^2)^2 + (\lambda\alpha)^2}\cos(\alpha t).$$

Since

$$\begin{aligned} X\cos(\alpha t) + Y\sin(\alpha t) &= \sqrt{X^2+Y^2}\Bigg[\underbrace{\frac{X}{\sqrt{X^2+Y^2}}}_{\sin\phi}\sin(\alpha t) + \underbrace{\frac{Y}{\sqrt{X^2+Y^2}}}_{\cos\phi}\cos(\alpha t)\Bigg] \\ &= \sqrt{X^2+Y^2}(\sin\phi\sin(\alpha t) + \cos\phi\cos(\alpha t)) \\ &= \sqrt{X^2+Y^2}\cos(\alpha t - \phi), \end{aligned}$$

the amplitude of $x_p(t)$ is

$$\sqrt{C^2 + D^2} = \sqrt{\frac{(a\lambda a)^2 + a^2(\omega^2-\alpha^2)^2}{[(\omega^2-\alpha^2)^2 + (\lambda\alpha)^2]}} = \frac{a}{\sqrt{(\omega^2-\alpha^2)^2+(\lambda\alpha)^2}}.$$

Back to the question: here $a = 10^{0.1D-8}/3$, $\lambda = 3.07$, $\omega = 4150$, and $\alpha = 1320\pi$. We want to find D such that the amplitude of the oscillation is 1, i.e.,

$$\frac{10^{0.1D-8}}{3}\cdot\frac{1}{\sqrt{(4150^2 - 1320^2\pi^2)^2 + (3.07\cdot 1320\pi)^2}} = 1 \implies D \approx 129 \text{ dB}.$$

16 Higher Order Linear Equations

Problem 16.1.4

Find the general solution to

$$\frac{\mathrm{d}^4x}{\mathrm{d}t^2} - 5\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = e^t.$$

Solution

We first solve for the complementary function by "guessing" $x(t)$ is for form e^{kt}. Then the auxiliary equation gives

$$k^4 - 5k^2 + 4 = 0 \implies (k^2 - 4)(k^2 - 1) = 0.$$

Therefore the complementary function is

$$x_h(t) = Ae^{2t} + Be^{-2t} + Ce^t + De^{-t}.$$

For the particular integral, since e^t is already part of the complementary solution we "guess" it's now of form $x_p(t) = Kte^t$. Then

$$\dot{x}_p = K(t+1)e^t,\ \ddot{x}_p = K(t+2)e^t,\ \dddot{x}_p = K(t+3)e^t,\ \ddddot{x}_p = K(t+4)e^t.$$

Therefore the original equation becomes

$$[K(t+4) - 5K(t+2) + 4Kt]e^t = e^t \implies K = -\frac{1}{6}$$

and the general solution is

$$x(t) = Ae^{2t} + Be^{-2t} + Ce^t + De^{-t} - \frac{te^t}{6}.$$

Problem 16.2 (partial)

Deduce that, for f_1, f_2, and f_3 on an interval I, if the Wronskian

$$W[f_1, f_2, f_3](t) := \begin{vmatrix} f_1 & f_2 & f_3 \\ \frac{\mathrm{d}f_1}{\mathrm{d}t} & \frac{\mathrm{d}f_2}{\mathrm{d}t} & \frac{\mathrm{d}f_3}{\mathrm{d}t} \\ \frac{\mathrm{d}^2 f_1}{\mathrm{d}t^2} & \frac{\mathrm{d}^2 f_2}{\mathrm{d}t^2} & \frac{\mathrm{d}^2 f_3}{\mathrm{d}t^2} \end{vmatrix}$$

is nonzero for some $t \in I$ then f_1, f_2, and f_3 are linearly independent.

Solution

Denote the matrix by $W(t)$ so that $\det[W(t)] = W[f_1, f_2, f_3](t)$. Suppose $W[f_1, f_2, f_3](t_0) \neq 0$. Then the matrix $W(t_0)$ is invertible. Solving $W(t_0)\boldsymbol{\alpha} = \mathbf{0}$ gets easy:

$$W(t_0)\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \implies \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = [W(t_0)]^{-1}\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Therefore

$$\begin{bmatrix}f_1 & f_2 & f_3\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = 0 \implies \alpha_1 = \alpha_2 = \alpha_3 = 0,$$

and thus f_1, f_2, and f_3 are linearly independent.