MATH 580 Problem Set 1

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Problem 1

Show that if E is a Hamel basis of a vector space V then every $v \in V$ has a unique representation as a linear combination of elements of E. Moreover, show that if E has $n \in \mathbb{N}$ elements then

(1) every basis of V has n elements, and

(2) every linearly independent set consisting of n elements is a basis of V.

Solution

Suppose, for contradiction, that for some $v \in V$ we have

$$v = \sum_{i \in I_1} \alpha_i e_i = \sum_{i \in I_2} \beta_i e_i \text{ with } \alpha_i, \beta_i \in \mathbb{K}, e_i \in E$$

for some different index sets $I_1, I_2 \subset \mathbb{N}$, i.e., v can expressed as two different linear combinations of elements of E. Now if we define $I = I_1 \cup I_2$,

$$\alpha_i^* := \begin{cases} \alpha_i & \text{if } i \in I_1 \\ 0 & \text{otherwise} \end{cases}, \text{ and } \beta_i^* := \begin{cases} \beta_i & \text{if } i \in I_2 \\ 0 & \text{otherwise} \end{cases},$$

we get

$$\sum_{i \in I_1} \alpha_i e_i = \sum_{i \in I} \alpha_i^* e_i \text{ and } \sum_{i \in I_2} \beta_i e_i = \sum_{i \in I} \beta_i^* e_i.$$

It follows that

$$0 = v - v = \sum_{i \in I} \alpha_i^* e_i - \beta_i^* e_i = \sum_{i \in I} (\alpha_i^* - \beta_i^*) e_i.$$

Since these e_i's are from E, they are still linearly independent, and hence $\alpha_i^* = \beta_i^*$ for all $i \in I$. Therefore $x \in I_1 \implies \alpha_i \neq 0 \implies \alpha_i^* \neq 0 \implies \beta_i^* \neq 0 \implies i \in I_2$, and likewise $i \in I_2 \implies I \in I_1$. Hence $I_1 = I_2$ and $\alpha_i = \beta_i$. Thus v can only be spanned by a specific subset of E with specific coefficients.

To show that if $\dim(V) = n < \infty \implies$ basis has n elements, we assume, for contradiction, that $W = \{w_i\}$ is a basis with $m \neq n$ elements. Since no other restrictions are imposed on E and W, WLOG assume $m > n$. Since E spans V, each $w_i \in W$ can be written as a linear combination of e_i's. Now suppose we want to solve the equation $\sum_{i=1}^{m} \beta_i w_i = 0$. With E, we can rewrite this as

$$\begin{bmatrix} \beta_1 & \cdots & \beta_m \end{bmatrix} \begin{bmatrix} M_{m\times n} \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = 0.$$

The middle matrix is short and wide, so its nullspace has dimension $\geqslant m-n > 0$ and thus the whole equation has a nontrivials solution. This contradicts e_i's being linearly independent! Hence $m > n$ is impossible. Likewise, $n > m$ is impossible, so we conclude that $m = n$.

Suppose we have linearly independent $\{v_1, \ldots, v_n\}$ but no linear combination of this set produces $v^* \in V$. Then the new set $\{v_1, \ldots, v_n, v^*\}$ is again linearly independent and it consists of $n+1$ elements. This is clearly a contradiction by the part above.

Problem 2

Show that if E is a linearly independent subset of a vector space V then there exists a Hamel basis of V containing E.

Solution

If $\dim(V) = n < \infty$ then we can extend E to a basis. If E is already a basis then we are done. Otherwise there exists some $v_1 \in V$ that cannot be expressed as a linear combination of elements of E. Hence $E \cup \{v_1\}$ is again linearly independent. If this new set is a basis then we are again done; if not there exists some $v_2 \in E \setminus \operatorname{span}(E \cup \{v_1\})$. Then $E \cup \{v_1, v_2\}$ is again linearly independent. We can proceed all the way till this extended set contains precisely n elements. By the lemma we know a linearly independent set containing n elements has to be a basis, and we are done.

†(This proof also works for finite-dimensional V's, so the previous paragraph becomes redundant.) Now suppose V is infinite-dimensional. We know it has a Hamel basis E^*. Define $\mathcal{E} := E \cup E^*$. Let $P := \{F \subset V : E \subset F \subset \mathcal{E}, F \text{ linearly independent}\}$ and $\subset$ as a partial order.

Claim: any arbitrary chain $C \subset P$ has an upper bound defined by $\mathcal{F} := \bigcup_{F_i \in C} F_i$. Clearly $E \subset \mathcal{F} \subset \mathcal{E}$. On one hand, if we take any finite collection of elements of $\mathcal{F}$, it entirely belongs to some $F_i \in C$ which is linearly independent. Hence $\mathcal{F}$ is linearly independent and $\mathcal{F} \in P$. On the other hand, $F_i \subset \mathcal{F}$ for all F_i's, so $\mathcal{F}$ is indeed an upper bound of C.

Going back to the main proof: since each chain has an upper bound, by Zorn's lemma there exists a maximal element $E_{\max} \in P$. A maximally linearly independent subset of V must be a Hamel basis, and we are done.

Problem 3

Show that $T \in L(X,Y)$ is injective if and only if $\ker(T) = \{0\}$.

Solution

For $\Longrightarrow$, assume T is injective. Suppose $\ker(T) \neq \{0\}$, then $T(x_0) = 0$ for some nonzero $x_0 \in X$. Pick any $x_1 \in X$ with $T(x_1) = (y_1)$. By linearity we get $T(x_1 + x_0) = T(x_1) + T(x_0) = T(x_1)$ whereas $x_1 + x_0 \neq x_1$. This contradicts T's injectivity.

For $\Longleftarrow$, assume $\ker(T) = \{0\}$ and suppose $T(x_1) = T(x_2)$ for $x_1, x_2 \in X$. It follows that

$$0 = T(x_1) - T(x_2) = T(x_1 - x_2) \implies x_1 - x_2 \in \ker(T) \implies x_1 - x_2 = 0.$$

Problem 4

Show that if $T \in L(X,Y)$ then both $\ker(T)$ and $\operatorname{im}(T)$ are vector spaces.

Solution

Suppose $x_1, x_2 \in \ker(T)$ and $\lambda \in \mathbb{K}$. Then $T(x_1 + x_2) = T(x_1) + T(x_2) = 0 \implies x_1 + x_2 \in \ker(T)$. Also, $T(\lambda x_1) = \lambda T(x_1) = 0$. Hence $\ker(T)$ is a vector space.

On the other hand, suppose $y_1, y_2 \in \operatorname{im}(T) \subset Y$ and $y_1 = T(x_1), y_2 = T(x_2)$ for some $x_1, x_2 \in X$. Then

$$y_1 + y_2 = T(x_1) + T(x_2) = T(x_1 + x_2) \implies y_1 + y_2 \in \operatorname{im}(T),$$

and

$$\lambda y_1 = \lambda T(x_1) = T(\lambda x_1) \implies \lambda y_1 \in \operatorname{im}(T).$$

Hence $\operatorname{im}(T)$ is a vector space.

Problem 5

(1) Show that $m^p \leqslant \sum_{i=1}^n |x_i|^p \leqslant n m^p$ for any $n \in \mathbb{N}$ and $p \in [1, \infty)$, where $m := \max_{i=1,\ldots,n} |x_i|$.

Solution

Denote m by $|x_k|$. Then the inequalities are immediate since

$$m^p = |x_k|^p \leqslant \sum_{i \neq k} |x_i|^p + |x_k|^p \leqslant \sum_{i=1}^{n} |x_k|^p = nm^p.$$

(2) Deduce that for each n

$$\lim_{p \to \infty} \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} = m.$$

Solution

Immediate by squeeze theorem: $\lim_{p \to \infty} (m^p)^{1/p} = m$ and

$$\lim_{p \to \infty} (nm^p)^{1/p} = m \lim_{p \to \infty} \left[\sqrt[p]{n} \right] = m.$$

It follows that $\lim_{p \to \infty} (\sum_{i=1}^{n} |x_i|^p)^{1/p}$ must also be m.

(3) Show that if $x \in \ell^1$ and $\epsilon > 0$ then there exists $N \in \mathbb{N}$ such that, for all p, we have

$$\|x\|_{\ell^p} - \epsilon \leqslant \|(x_1, \ldots, x_n)\|_{\ell^p} \leqslant \|x\|_{\ell^p} \text{ for all } n \geqslant N.$$

Solution

First thing to notice is that $\|(x_1,\dots,x_n)\|_{\ell^1} \coloneqq \sum_{i=1}^n |x_i| \uparrow \sum_{i\geqslant 1}|x_i| = \|x\|_{\ell^1}$ as $n\to\infty$. Therefore the sequence $\{a_n\}$ defined by $a_n \coloneqq \|(x_1,\dots,x_n)\|_{\ell^p}$ is a monotone increasing sequence that converges to $\|x\|_{\ell^p}$.

Clearly when $p=1$ the problem's statement holds since x needs to be an absolutely convergent sequence. Hence, given $\epsilon>0$, there exists $N\in\mathbb{N}$ such that

$$n\geqslant N \implies \|x\|_{\ell^1}-\epsilon \leqslant \|(x_1,\dots,x_n)\|_{\ell^1} \leqslant \|x\|_{\ell^1},$$

i.e.,

$$\|x\|_{\ell^1} - \|(x_1,\dots,x_n)\|_{\ell^1} = \sum_{i\geqslant 1}|x_i| - \sum_{i=1}^n |x_i| = \sum_{i\geqslant n+1}|x_i| \leqslant \epsilon.$$

If $p>1$ we have $\|x\|_{\ell^p}\leqslant \|x\|_{\ell^1}$. Then it follows that

$$\begin{aligned}\|x\|_{\ell^p} - \|(x_1,\dots,x_n)\|_{\ell^p} &= \|x\|_{\ell^p} - \|(x_1,\dots,x_n,0,\dots)\|_{\ell^p}\\ &\leqslant \|(0,\dots,x_{n+1},x_{n+2},\dots)\|_{\ell^p} && \text{(triangle inequality)}\\ &\leqslant \|(0,\dots,x_{n+1},x_{n+2},\dots)\|_{\ell^1}\\ &= \sum_{i\geqslant n+1}|x_i| \leqslant \epsilon,\end{aligned}$$

and so

$$n\geqslant N, p\geqslant 1 \implies \|x\|_{\ell^p}-\epsilon \leqslant \|(x_1,\dots,x_n)\|_{\ell^p} \leqslant \|x\|_{\ell^p},$$

as desired.

(4) Deduce that if $x\in\ell^1$ then $\|x\|_{\ell^\infty} = \lim_{p\to\infty}\|x\|_{\ell^p}$.

Solution

By (2) we know $\|(x_1,\dots,x_n)\|_{\ell^p}$ converges in p, and by (3) we see that it actually converges uniformly with respect to p [whatever works for $p=1$ works for $p>1$ as well]. Clearly $\|(x_1,\dots,x_n)\|_{\ell^p}$ converges with respect to n too. Then, by Moore-Goode Theorem* we have

$$\lim_{p\to\infty}\|x\|_{\ell^p} = \lim_{p\to\infty}\left[\lim_{n\to\infty}\|(x_1,\dots,x_n)\|_{\ell^p}\right] \overset{*}{=} \lim_{n\to\infty}\left[\max_{1\leqslant i\leqslant n}|x_i|\right] = \sup_{i\geqslant 1}|x_i| = \|x\|_{\ell^\infty}.$$

Problem 6

Give another proof of the Hölder's inequality (i.e., not using Jensen's),

$$\|fg\|_{L^1} \leqslant \|f\|_{L^p}\|g\|_{L^q} \text{ where } p^{-1}+q^{-1}=1,$$

and prove the Minkowski's inequality, i.e.,

$$\|f+g\|_{L^p} \leqslant \|f\|_{L^p}+\|g\|_{L^p}.$$

Proof: Hölder

First normalize f and g by defining $F := |f|/\|f\|_{L^p}$ and $G := |g|/\|g\|_{L^p}$. It follows that $\|F\|_{L^p} = \|G\|_{L^p} = 1$. By the convexity of exponential function we have

$$\exp\left(\frac{x}{p}+\frac{y}{q}\right) \leqslant \frac{e^x}{p}+\frac{e^y}{q} \text{ where } \frac{1}{p}+\frac{1}{q}=1.$$

Letting $x := \ln(F^p)$ and $y := \ln(G^q)$ gives

$$\exp\left(\frac{x}{p}+\frac{y}{q}\right) = \exp\left(\ln(FG)\right) = FG \leqslant \frac{F^p}{p}+\frac{G^q}{q},$$

i.e.,

$$\frac{|fg|}{\|f\|_{L^p}\|g\|_{L^p}} \leqslant \frac{|f|^p}{p\|f\|_{L^p}}+\frac{|g|^q}{q\|g\|_{L^q}}.$$

Integrating gives

$$\int_\Omega FG\,\mathrm{d}\mu = \frac{1}{\|f\|_{L^p}\|g\|_{L^q}}\int_\Omega |fg|\,\mathrm{d}\mu = \frac{\|fg\|_{L^1}}{\|f\|_{L^p}\|g\|_{L^q}}$$

and

$$\int_\Omega \frac{F^p}{p}+\frac{G^q}{q}\,\mathrm{d}\mu = \frac{1}{p}\int_\Omega F^p\,\mathrm{d}\mu+\frac{1}{q}\int_\Omega G^q\,\mathrm{d}\mu = \frac{1}{p}+\frac{1}{q}=1.$$

Hence

$$\frac{\|fg\|_{L^1}}{\|f\|_{L^p}\|g\|_{L^q}} \leqslant 1 \implies \|fg\|_{L^1} \leqslant \|f\|_{L^p}\|g\|_{L^q},$$

and we've proven Hölder's inequality without using Jensen's inequality.

Proof: Minkowski

First we need to verify whether $f+g \in L^p(\Omega)$:

$$\|f+g\|_{L^p}^p = \int_\Omega |f+g|^p\,\mathrm{d}\mu \leqslant \int_\Omega \left[2\max\{|f|,|g|\}\right]^p\,\mathrm{d}\mu \leqslant 2^q\int_\Omega |f|+|g|\,\mathrm{d}\mu = 2^q(\|f\|_{L^p}^p+\|g\|_{L^q}^q) < \infty.$$

Now back to proving Minkowski's inequality. If $p=1$ then by triangle inequality $|f+g| \leqslant |f|+|g|$ we have

$$\int_\Omega |f+g|\,\mathrm{d}\mu \leqslant \int_\Omega |f|+|g|\,\mathrm{d}\mu \implies \|f+g\|_{L^1} \leqslant \|f\|_{L^1}\|g\|_{L^1}.$$

If $p = \infty$ then

$$S_1 := \{x \in \Omega : |f(x)| > \|f\|_{L^\infty}\} \text{ and } S_2 := \{x \in \Omega : |g(x)| > \|g\|_{L^\infty}\}$$

are null sets and so is their union. Hence

$$|f(x) + g(x)| \leqslant |f(x)| + |g(x)| \implies \|f+g\|_{L^\infty} \leqslant \|f\|_{L^\infty} + \|g\|_{L^\infty}.$$

In the most common case where $p \in (1, \infty)$, we have

$$\begin{aligned} \|f+g\|_{L^p}^p = \int_\Omega |f+g|^p \, \mathrm{d}\mu &= \int_\Omega |f+g|^{p-1}|f+g| \, \mathrm{d}\mu \\ &\leqslant \int_\Omega |f+g|^{p-1}(|f|+|g|) \, \mathrm{d}\mu \\ &= \int_\Omega |f+g|^{p-1}|f| \, \mathrm{d}\mu + \int_\Omega |f+g|^{p-1}|g| \, \mathrm{d}\mu \\ \text{(Hölder's)} &\leqslant (\|f\|_{L^p} + \|g\|_{L^p})\||f+g|^{p-1}\|_{L^q} \\ &= (\|f\|_{L^p}\|g\|_{L^q}) \left(\int_\Omega |f+g|^{(p-1)q} \, \mathrm{d}\mu\right)^{1/q}. \end{aligned}$$

The last integral evaluates to 0 if and only if f and g are both 0 μ-a.e. If that is the case, then Minkowski's reduces to $0 \leqslant 0$, trivial. Otherwise, we may divide by it. Since $(p-1)q = (p-1) \cdot p/(p-1) = p$, we get

$$\frac{\int_\Omega |f+g|^p \, \mathrm{d}\mu}{\left(\int_\Omega |f+g|^{(p-1)q} \, \mathrm{d}\mu\right)^{1/q}} = \left(\int_\Omega |f+g|^p \, \mathrm{d}\mu\right)^{1/p}.$$

Then it follows that

$$\left(\int_\Omega |f+g|^p \, \mathrm{d}\mu\right)^{1/p} = \|f+g\|_{L^p} \leqslant \|f\|_{L^p} + \|g\|_{L^p},$$

hence we've shown Minkowski's inequality.

Problem 7

Show that if $\mu(\Omega) < \infty$ then $L^q(\Omega) \subset L^p(\Omega)$ whenever $p \leqslant q$. Moreover, given $p \in [1, \infty]$ find a function $f \in L^p(\mathbb{R}^n)$, with Lebesgue measure, such that $f \notin L^q(\mathbb{R}^n)$ for any $q \neq p$. [Consider $f(x) := |x|^{-n/p}(1 + \ln^2|x|)^{-1/p}$ for $p < \infty$.] *This shows that there is no inclusion $L^p \subset L^q$ when $\Omega := \mathbb{R}^n$.*

Solution

The first part can be shown using Hölder's inequality. Suppose $f \in L^q(\Omega)$, i.e., $\|f\|_{L^q} < \infty$. Define $g := 1$.

The case $p = q$ is trivial; if not, by Hölder's inequality we have

$$\begin{aligned}\|f\|_{L^p}^p &= \int_\Omega |f|^p \cdot 1 \,\mathrm{d}\mu \\ &\leqslant \left(\int_\Omega (|f|^p)^{q/p} \,\mathrm{d}\mu\right)^{p/q} \cdot \left(\int_\Omega 1 \,\mathrm{d}\mu\right)^{1-p/q} \\ &= \left(\int_\Omega |f|^q \,\mathrm{d}\mu\right)^{p/q} \cdot \left(\int_\Omega 1 \,\mathrm{d}\mu\right)^{1-p/q} \\ &= \|f\|_{L^q}^p \cdot \mu(\Omega)^{1-p/q} \\ &< \infty.\end{aligned}$$

Hence $f \in L^q(\Omega) \implies f \in L^p(\Omega)$, i.e., $L^q(\Omega) \subset L^p(\Omega)$.

For the second part, if $p = \infty$ simply define $f := 1$. Clearly $\|f\|_{L^\infty} = 1$ whereas $\|f\|_{L^p} = \mu(\mathbb{R}^n) = \infty$ whenever $x \in [1, \infty)$. For $p < \infty$, let $f(x) := |x|^{(-n/p)}(1 + \ln^2|x|)^{(-1/p)}$. Then

$$\begin{aligned}\int_\Omega |f|^p \,\mathrm{d}\mu &= \int_{\mathbb{R}^n} |f|^p \,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \frac{1}{|x|^n(1+\ln^2|x|)} \,\mathrm{d}x \\ &\overset{\dagger}{=} \int_{S^{n-1}} \int_0^\infty \frac{1}{r^n(1+\ln^2(r))} r^{n-1} \mathrm{d}r \,\mathrm{d}\theta \\ &= \int_{S^{n-1}} \int_0^\infty \frac{1}{r(1+\ln^2(r))} \,\mathrm{d}r \,\mathrm{d}\theta \\ &= S_{n-1}\left[\arctan(\ln(r)) + C\right]\Big|_{r=0}^{\infty} \\ &= \pi S_{n-1}\left(= \frac{2\pi^{(n+1)/2}}{\Gamma(n/2)}\right) < \infty\end{aligned}$$

and so $f \in L^p(\mathbb{R}^n)$.

Now if $q \neq p$, we have

$$\begin{aligned}\int_\Omega |f|^q \,\mathrm{d}\mu &= S_{n-1} \int_0^\infty \frac{r^{n-1}}{r^{nq/p}(1+\ln^2(r))^{q/p}} \,\mathrm{d}x \\ &\sim \int_0^\infty \frac{1}{r^{nq/p-(n-1)}(1+\ln^2(r))^{q/p}} \,\mathrm{d}r.\end{aligned}$$

If $q < p$, then $nq/p - (n-1) < 1$ and there exists $\epsilon > 0$ satisfying $nq/p - (n-1) + \epsilon < 1$ and $\epsilon \neq 1$ [used in L'Hôpital later]. Then,

$$\begin{aligned}\int_0^\infty \ldots \,\mathrm{d}r &> \int_1^\infty \ldots \,\mathrm{d}r \\ &= \int_1^\infty \frac{1}{r^{nq/p-(n-1)+\epsilon}} \cdot \frac{r^{nq/p-(n-1)+\epsilon}}{r^{nq/p-(n-1)}(1+\ln^2(r))^{q/p}} \,\mathrm{d}r.\end{aligned}$$

Notice that, since $nq/p - (n-1) + \epsilon < 1$, the integral $\int_0^\infty 1/r^{nq/p-(n-1)+\epsilon}\,\mathrm{d}r$ diverges. On the other hand, applying L'Hôpital's rule to the second term twice gives

$$\begin{aligned}\lim_{r\to\infty} \frac{r^{nq/p-(n-1)+\epsilon}}{r^{nq/p-(n-1)}(1+\ln^2(r))^{q/p}} &= \lim_{r\to\infty} \frac{r^{\epsilon}}{(1+\ln^2(r))^{q/p}} \\ &> \lim_{r\to\infty} \frac{r^{\epsilon}}{1+\ln^2(r)} \left(= \frac{\infty}{\infty}\right) \\ &\overset{H}{=} \lim_{r\to\infty} \frac{\epsilon r^{\epsilon}}{2\ln r} \left(= \frac{\infty}{\infty}\right) \\ &\overset{H}{=} \lim_{r\to\infty} \frac{\epsilon^2 r^{\epsilon}}{2} = \infty.\end{aligned}$$

Hence $\int_1^\infty \ldots\,\mathrm{d}r = \infty$ and $f \notin L^q(\mathbb{R}^n)$ if $q < p$.

If $\infty > q > p$, we can find $0 < \epsilon < 1$ such that $nq/p - (n-1) - \epsilon > 1$. Then

$$\begin{aligned}\int_0^\infty \ldots\,\mathrm{d}r &= \int_0^\infty \frac{1}{r^{nq/p-(n-1)-\epsilon}} \cdot \frac{r^{nq/p-(n-1)-\epsilon}}{r^{nq/p-(n-1)}(1+\ln^2(r))^{q/p}}\,\mathrm{d}r \\ &< \int_0^1 \frac{1}{r^{nq/p-(n-1)-\epsilon}}\,\mathrm{d}r \int_0^1 \frac{r^{-\epsilon}}{(1+\ln^2(r))^{q/p}}\,\mathrm{d}r.\end{aligned}$$

The first integral diverges since $\lim_{x\downarrow 0} \frac{1}{r^{nq/p-(n-1)-\epsilon}} = \infty$. The second integral also diverges since

$$\begin{aligned}\lim_{x\downarrow 0} \frac{r^{-\epsilon}}{(1+\ln^2(r))^{q/p}} \sim \lim_{x\downarrow 0} \frac{r^{-\epsilon}}{(\ln^2(r))^{q/p}} &> \lim_{x\downarrow 0} \frac{r^{-\epsilon}}{(\ln^2(r))^{q}} \\ &= \lim_{x\downarrow 0} \frac{r^{-\epsilon}}{(\ln(r))^{2q}} \left(= \frac{\infty}{\infty}\right) \\ &\underbrace{\overset{H}{=} \ldots \overset{H}{=}}_{2q \text{ times}} \lim_{x\downarrow 0} \frac{(-\epsilon)^{2q} r^{-\epsilon}}{(2q)!} = \infty.\end{aligned}$$

Hence $\int_0^1 \ldots\,\mathrm{d}r$ diverges and $f \notin L^q(\mathbb{R}^n)$ if $\infty > q > p$, either.

If $q = \infty$, simply notice that

$$\begin{aligned}\lim_{x\to 0} |f|^p = \lim_{r\downarrow 0} \frac{1}{r^n(1+\ln^2(r))} &\sim \lim_{r\downarrow 0} \frac{r^{-n}}{\ln^2(r)} \\ &\overset{H}{=} \lim_{r\downarrow 0} \frac{-nr^{-n}}{2\ln(r)} \\ &\overset{H}{=} \lim_{r\downarrow 0} \frac{n^2 r^{-n}}{2} = \infty.\end{aligned}$$

This means $\lim_{x\to 0}|f| = \infty$ as well, and so $\operatorname{ess\,sup}|f| = \|f\|_{L^\infty} = \infty$, i.e., $f \notin L^\infty(\mathbb{R}^n)$.

Problem 8

Show that $\|\cdot\|_{L^p}$ is a norm on $C([0,1])$ for any $p \in [1, \infty]$, and that $\|\cdot\|_{L^\infty}$ is the same as the "maximum norm" $\|f\|_{\max} := \max_{[0,1]}|f|$. Moreover, show that if $f, f_n \in C([0,1])$ for every $n \geqslant 1$ and $\|f_n - f\|_{\max} \to 0$, i.e., f_n converges "uniformly" to f, then $\|f_n - f\|_{L^p(0,1)} \to 0$ for every p, i.e., $f_n \to f$ "in L^p" and that $f_n(x) \to f(x)$ for every $x \in [0,1]$, i.e., $f_n \to f$ "pointwise".

Solution

For $p < \infty$, we show $\|\cdot\|_{L^p}$ defines a norm by definition:

(1) Non-degeneracy: $\int_0^1 |f|^p \,\mathrm{d}x \geqslant 0 \implies \|f\|_{L^p}^p \geqslant 0 \implies \|f\|_{L^p} \geqslant 0$. Furthermore, $\|f\|_{L^p} = 0 \iff \int_0^1 |f|^p \,\mathrm{d}x = 0 \iff |f| = 0$ for all $x \in [0,1]$ (since f is continuous).

(2) Absolute homogeneity:

$$\|\lambda f\|_{L^p} = \left(\int_0^1 |\lambda f|^p \,\mathrm{d}x\right)^{1/p} = \left(|\lambda|^p \int_0^1 |f|^p \,\mathrm{d}x\right)^{1/p} = |\lambda| \|f\|_{L^p}.$$

(3) Triangle inequality: immediate from Minkowski's inequality.

Hence $\|\cdot\|_{L^p}$ defines a norm on $C([0,1])$.

Since $|f|$ is continuous and $[0,1]$ compact, it attains maximum at $|f(m)|$ for some $m \in [0,1]$. If the maximum is 0 then $f \equiv 0$ and obviously $\operatorname*{ess\,sup}_{[0.1]}|f| = 0 = \max_{[0,1]}|f|$. Otherwise $|f(m)| > 0$. By the continuity of $|f|$, given $\epsilon > 0$ there exists $\delta > 0$ satisfying $|f(m)| - \epsilon < |f(n)| < |f(m)|$ for all $n \in (m - \delta, m + \delta) \cap [0,1]$. Clearly this interval has positive measure, so

$$\operatorname*{ess\,sup}_{[0,1]}|f| = \inf\{b \in \mathbb{R} : \mu(\{x : f(x) > b\}) = 0\} \geqslant |f(m)|.$$

On the other hand $|f(m)|$ does satisfy this condition. Hence $\|f\|_{L^\infty} = \operatorname*{ess\,sup}_{[0,1]}|f| = |f(m)| = \max_{[0,1]}|f| = \|f\|_{\max}$.

Now assume $p < \infty$ and f_n converges "uniformly" to f (if $p = \infty$ this entire statement becomes since $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{\max}$ are equivalent, as shown above). It follows that

$$\begin{aligned}
\|f_n - f\|_{L^p} &= \left(\int_0^1 |f_n - f|^p \,\mathrm{d}x\right)^{1/p} \\
&\leqslant \left(\int_0^1 \max_{[0,1]}|f_n - f|^p \,\mathrm{d}x\right)^{1/p} \\
&= \|f_n - f\|_{\max} \to 0,
\end{aligned}$$

and

$$|f_n(x) - f(x)| \leqslant \max_{[0.1]}|f_n - f| \to 0 \text{ for all } x \in [0,1].$$

Problem 9

Let $(X, \|\cdot\|)$ be a normed space and $A \subset B \subset X$. Show that if A is dense in B and B is dense in X then A is dense in X.

Proof

Let $\epsilon > 0$ be given. Pick any $x \in X$. It follows that there exists $b \in B$ such that $\|b - x\| < \epsilon/2$. Simlarly, there exists $a \in A$ such that $\|a - b\| < \epsilon/2$. Therefore by triangle inequality we have

$$\|a - x\| \leqslant \|a - b\| + \|b - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence A is dense in X.

Problem 10

Show that if U is an open subspace of a normed space X then $U = X$.

Proof

We need to show mutual inclusion between U and X, and $U \subset X$ is trivial. Hence it suffices to show $X \subset U$.

Clearly $0 \in U$ (the additive identity). Then there exists $r > 0$ such that $B(0, r) \subset U$. Pick any $x \in X$, and define† $x' := \frac{r}{2} \cdot \frac{x}{\|x\|}$. Then we have

$$\|x'\| = \left\| \frac{r}{2} \cdot \frac{x}{\|x\|} \right\| = \frac{r}{2} \implies x' \in B(0, r) \implies x' \in U.$$

Since U is a subspace, $x' \in U \implies r/(2\|x\|) \cdot x' = x \in U$. Hence $X \subset U$ and we've shown $U = X$.