MATH 580 Problem Set 2

Qilin Ye

December 15, 2020

Problem 1

(1) Show that c_0 equipped with the ℓ^∞ norm is separable.

(2) Show that $L^\infty(0,1)$ is not separable.

Solution

(1) To show c_0 is separable, it suffices to show that c_0 has a countable dense subset. Claim: $c_{00} \cap \ell^\infty(\mathbb{Q})$, i.e., sequences in c_{00} with rational entries, is dense in c_0.

Let $x := \{x_n\}_{n\geqslant 1} \in c_0$ and $\epsilon > 0$ be given. We want to show there exists $x' := \{x'_n\}_{n\geqslant 1} \in c_{00} \cap \ell^\infty(\mathbb{Q})$ such that $\|x - x'\|_{\ell^\infty} < \epsilon$. By the convergence of x, there exists $N \in \mathbb{N}$ such that $|x_n| < \epsilon$ whenever $n > N$. If we let $x' := \{x_1, \ldots, x_n, 0, \ldots\}$ then

$$\|x - x'\|_{\ell^\infty} = \sup_{i\geqslant 1} |x_i| = \sup_{i>n} |x_i| < \epsilon,$$

which completes the proof of c_0's separability.

(2) Similar to the proof of Example 2.11.3, we want to find an uncountable set S such that $\|x - y\|_{L^\infty} = 1$ for all distinct $x, y \in S$. Consider the following partition of $(0,1)$:

$$\begin{aligned}(0,1) &= \bigcup_{i\geqslant 1}\left[\left(1 - \frac{1}{2^{i-1}}, 1 - \frac{1}{2^i}\right) \cup \left\{1 - \frac{1}{2^i}\right\}\right] \\ &= \underbrace{\left(0, \frac{1}{2}\right)}_{=:I_1} \cup \underbrace{\left(\frac{1}{2}, \frac{3}{4}\right)}_{=:I_2} \cup \cdots \cup \underbrace{\left\{\frac{1}{2}, \frac{3}{4}, \ldots\right\}}_{D}.\end{aligned}$$

Notice that all the (countably infinite) I_n's are intervals which have positive measure whereas the last set D is countable and thus null. If we define

$$\mathcal{S} := \{f \in L^\infty(0,1)\} \text{ where } f(x) = \begin{cases} 0 & \text{if } x \in D \\ 0 \text{ or } 1 & \text{if } x \in I_n, \text{ but } f \text{ remains constant within the same } I_n \end{cases}$$

It becomes clear that there exists a bijection between $\mathcal{S}$ and $E := \{x \in \ell^\infty : x_i \in \{0,1\}\}$, our example in class. Furthermore, if $f_1 \neq f_2$ then there exists some I_n such that $|f_1(x) - f_2(x)| = 1$ for all $x \in I_n$. Therefore $\|f_1 - f_2\|_{L^\infty} = 1$ for all distinct $f_1, f_2 \in L^\infty(0,1)$. Now it just follows from the same argument: if we approximate $L^\infty(0,1)$ using any dense subset A we have $|A| \geqslant |\mathcal{S}|$, since, given $f \in S$, there should exist $g \in A$ with $\|g - f\|_{L^\infty} < 1/2$, but the $1/2$–neighborhoods of different f's are disjoint by Minkowski's inequality, which means distinct elements of $\mathcal{S}$ can only be approximated by distinct elements of A.

Problem 2

Let X, Y be normed spaces and $T \in L(X,Y)$. Show that

$$\sup_{\|x\|_X = 1} \|T(x)\|_Y = \sup_{\|x\|_X \leqslant 1} \|T(x)\|_Y = \sup_{x \neq 0} \frac{\|T(x)\|_Y}{\|x\|_X}.$$

This shows the characterization of the operator norm $\|T\|_{B(X,T)}$; recall Lemma 2.14.

Solution

The first inequality comes from the fact that, if $\|x\|_X < 1$ then (in Y)

$$\|T(x)\| = \left\|T\left(\frac{x}{\|x\|}\right)\|x\|\right\| = \|x\|\left\|T\underbrace{\left(\frac{x}{\|x\|}\right)}_{\|\cdot\|_X = 1}\right\| < \left\|T\left(\frac{x}{\|x\|}\right)\right\|,$$

so the supremum when $\|x\|_X \leqslant 1$ is the same as that when $\|x\|_X = 1$. The second equality is because (also directly from definition of $\|T\|$)

$$\frac{\|T(x)\|_Y}{\|x\|_X} = \left\|T\left(\frac{x}{\|x\|_X}\right)\right\| \text{ as long as } x \neq 0$$

so scaling/normalizing x will not affect the answer, i.e., $\sup_{\|x\|_X = 1} \|T(x)\|_Y = \sup_{x \neq 0} \frac{\|T(x)\|_Y}{\|x\|_X}$.

Problem 3

Show that if K is a compact subset of a normed space X then K is closed and bounded.

Proof

Closedness: let $\{x_n\} \subset K$ be such that $x_n \to x \in X$. By the compactness of K there exists a subsequence that converges to $x' \in K$. It follows that the convergent also converges to x', but since limits are unqiue, $x = x' \in K$ which shows K is closed.

Boundedness: suppoe K is not bounded, then for some $k \in K$ we are able to find a sequence $(k, y_1, y_2, \cdots)$ such that $\|k - y_i\| \geqslant i$. By triangle inequality, for any $m, n \in \mathbb{N}$ we have

$$\|k - y_n\| \leqslant \|k - y_m\| + \|y_m - y_n\| \implies \|k - y_n\| - \|k - y_m\| \leqslant \|x_n - x_m\|. \tag{Δ}$$

Suppose this sequence has a Cauchy subsequence and let $\epsilon = 1/2$ be given. Then there exists $N \in \mathbb{N}$ such that $\|y_m - y_n\| < \epsilon$ whenever $m > n > N$. Fix n and let m be such that $m \geqslant \|k - y_n\| + 1$. By Δ we have $\|y_n - y_m\| \geqslant \|k - y_m\| - \|k - x_n\| \geqslant 1 > \epsilon$, contradiction. Hence $(k, y_1, \cdots)$ has no Cauchy subsequence, in particular no convergent subsequence, but this contradicts the compactness of K.

Problem 4

(1) Let (Ω, μ) be a measure space. Show that $L^\infty(\Omega)$ with $\|\cdot\|_\infty$ is a Banach space.

(2) Show that c_0 is a closed subspace of ℓ^∞ and deduce that $(c_0, \|\cdot\|_{\ell^\infty})$ is a Banach space. You can use the fact that ℓ^p spaces for $p \in [1, \infty]$ are Banach.

Solution: Part 1

Let $\{f_n\} \subset L^\infty(\Omega)$ be Cauchy. That is, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|f_m - f_n\|_\infty < \epsilon/2$ whenever $m, n \geqslant N$. If we define

$$S_{m,n} := \{x \in \Omega : |f_m - f_n| > \|f_m - f_n\|_\infty\}$$

then by definition this is a null set. Since the union of countably many null sets is also null,

$$S := \bigcup_{m,n \geqslant N} S_{m,n}$$

is a null set. Notice that

$$S^c = \{x \in \Omega : |f_m - f_n| \leqslant \|f_m - f_n\|_\infty\}.$$

Now the remaining proof highly resembles that of $\mathcal{F}_b(\Omega; \mathbb{K}) := \{f : \Omega \to \mathbb{K} \mid f \text{ is bounded}\}$. For all $x \in S^c$ and for all $m, n \geqslant N$ we have

$$|f_m(x) - f_n(x)| \leqslant \|f_m - f_n\|_\infty < \frac{\epsilon}{2}.$$

In particular $\{f_n(x)\}$ is Cauchy for each $x \in S^c$. By the completeness of $\mathbb{K}$, $f(x) := \lim_{n\to\infty} f_n(x)$ is well defined for all $x \in S^c$. We simply need to let $f(x) :=$ anything, say 0, for $x \in S$.

Now it remains to show $f_m \to f \in L^\infty(\Omega)$. Fix m. By the convergence of $f_n(x)$, $|f_n(x) - f(x)| < \epsilon/2$ for sufficiently large n's. Therefore, for all $x \in S^c$,

$$|f_m(x) - f(x)| \leqslant |f_m(x) - f_n(x)| + |f_n(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and so taking supremum gives

$$\sup_{x\in S^c} |f_m(x) - f(x)| = \operatorname*{ess\,sup}_{x\in\Omega} |f_m - f| = \|f_m - f\|_\infty \leqslant \epsilon \implies f_m \to f.$$

It's also clear that $f \in L^\infty(\Omega)$ since, by triangle inequality,

$$\|f\|_\infty \leqslant \|f - f_m\|_\infty + \|f_m\|_\infty \leqslant \epsilon + \|f_m\|_\infty < \infty.$$

Solution: Part 2

Let $\{c^{(n)}\} \subset c_0$ be a sequence (of sequences) that converges to some $c \in \ell^\infty$ where $c^{(n)} := (c_1^{(n)}, c_2^{(n)}, \dots)$ is itself a sequence converging to 0 and $c := (c_1, c_2, \dots)$ bounded (i.e., in ℓ^∞). We want so show $c \in c_0$. Let $\epsilon > 0$ be given. By the convergence of $\{c^{(n)}\}$, there exists $N_1 \in \mathbb{N}$ such that $\|c^{(n)} - c\|_{\ell^\infty} < \epsilon/2$ whenever $n \geqslant N_1$. This means

$$\|c^{(n)} - c\|_{\ell^\infty} = \sup_{i\geqslant 1} |c_i^{(n)} - c_i| < \frac{\epsilon}{2} \implies |c_i - c_i^{(n)}| < \frac{\epsilon}{2} \text{ for all } i \geqslant 1.$$

Pick any $n \geqslant N_1$ and fix it. Since $c^{(n)} \in c_0$ the sequence $(c_1^{(n)}, c_2^{(n)}, \dots)$ converges to 0. Hence there exists $N_2 \in \mathbb{N}$ such that $|c_m^{(n)}| < \epsilon/2$ whenever $m \geqslant N_2$. By triangle inequality, we have

$$|c_i| \leqslant |c_i - c_i^{(n)}| + |c_i^{(n)}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ whenever } i \geqslant N_2,$$

which indeed shows $c_i \to 0$ as $i \to \infty$, i.e., $c \in c_0$. Hence c_0 is closed.

Assuming ℓ^∞ is Banach, by Lemma 3.20.2 we conclude that $(c_0, \|\cdot\|_{\ell^\infty})$ is also Banach.

Problem 5

Let X be a normed space and $U \subset X$ a closed subspace. The quotient space X/U is the set of all equivalent classes

$$[x] = x + U := \{x + u : u \in U\}$$

for $x \in X$. Show that

$$\|[x]\|_{X/U} := \inf_{u\in U} \|x + u\|_X \tag{*}$$

is a norm on X/U. If additionally X is Banach, show that X/U is Banach.

Solution: Part 1

For the first part (showing (*) defines a norm):

(1) Non-degeneracy: the non-degeneracy of $\|\cdot\|_X$ guarantees that $\|[x]\|_{X/U}$ will always be nonnegative. Now it remains to show that $\|[x]\|_{X/U} = 0 \iff [x] = [0]$.

For $\implies$, assume $\|[x]\|_{X/U} = 0$. Then there exists a sequence $\{u_n\} \subset U$ that converges to $-x \in X$ since $\|\cdot\|_X$ satisfies non-degeneracy. By the closedness of U we know $-x \in U$. Then $x = -(-x) \in U$. Since $x - 0 \in U$ we have $x \sim 0$, i.e., $[x] = [0]$.

For $\impliedby$, suppose $[x] = [0]$. Then since $0 \in U$, we have $\|[x]\|_{X/U} = \|[0]\|_{X/U} := \inf_{u \in U} \|0 + u\|_X = \|0\|_X = 0$.

(2) Absolute homogeneity: if $\lambda = 0$ then $\|\lambda[x]\|_{X/U} = \|[\lambda x]\|_{X/U} = \|0\|_{X/U} = 0 = \inf_{u \in U} \|0x + u\|_X$. Otherwise, it's easy to notice that $[x] = (\lambda[x])/\lambda = [\lambda x]/\lambda$ so $\|[x]\|_{X/U} = \inf_{u \in U} \|x + u/\lambda\|_X$. Then,

$$\begin{aligned}
\|\lambda[x]\|_{X/U} = \|[\lambda x]\|_{X/U} &= \inf_{u \in U} \|\lambda x + u\|_X \\
&= \inf_{u \in U} \|\lambda x + \lambda(u/\lambda)\|_X \\
&= |\lambda| \inf_{u \in U} \|x + u/\lambda\|_X \\
&= |\lambda| \|[x]\|_{X/U}, \text{ as desired.}
\end{aligned}$$

(3) Triangle inequality: similar to above, notice that $[x] = \inf_{u \in U} \|x + u\| = \inf_{u \in U} \|x + u/2\|$. Then,

$$\begin{aligned}
\|[x] + [y]\|_{X/U} = \|[x + y]\|_{X/U} &= \inf_{u \in U} \|x + y + u\| \\
&= \inf_{u \in U} \|x + y + 2u\| \\
&\leqslant \inf_{u \in U} (\|x + u\| + \|y + u\|) \\
&= \inf_{u \in U} \|x + u\| + \inf_{u \in U} \|y + u\| \\
&= \|[x]\|_{X/U} + \|[y]\|_{X/U}.
\end{aligned}$$

Hence $\|\cdot\|_{X/U}$ defines a norm on X/U.

Solution: Part 2

The proof of showing X/U is Banach relies only the fact that $[x] = [x+u]$ for all $u \in U$. Now let $\{[x_n]\} \subset X/U$ be a Cauchy sequence. Notice that this does *not* mean $\{x_n\}$ is Cauchy in X: the two spaces are equipped with different norms, and $\{[x_n]\}$ does not correspond to a unique $\{x_n\}$ since $[x] = [x+u]$. Nevertheless, we can use this property to come up with another sequence $\{[x_{n_k} - y_k]\}$ with $\{[y_n]\} \subset X/U$ (and the index of y comes from the index of the subsequence x_{n_k}) that is Cauchy in X. To avoid cumbersome notations we will denote $\|\cdot\|_{X/U}$ by $\|\cdot\|$ for the remaining of the problem.

Since $\{[x_n]\}$ is Cauchy, there exists a subsequence $\{[x_{n_k}]\}$ such that $\|[x_{n_k}]\| - \|[x_{n_{k+1}}]\| < 1/2^k$. Define $y_1 := 0 \in U$. It follows that

$$\|[x_{n_1} - y_1] - [x_{n_2}]\| = \|[x_{n_1}] - [x_{n_2}]\| = \|[x_{n_1} - x_{n_2}]\| = \inf_{u \in U} \|x_{n_1} - x_{n_2} + u\|_X < \frac{1}{2}.$$

Notice that we also have $\inf_{u \in U} \|x_{n_1} - x_{n_2} + u\|_X = \inf_{u \in U} \|x_{n_1} - y_1 - x_{n_2} + u\|_X$. Then there exists $y_2 \in U$ such that $\|(x_{n_1} - y_1) - (x_{n_2} - y_2)\|_X < 1/2$. Repeating the process, we have

$$\|[x_{n_2} - y_2] - [x_{n_3}]\| = \|[x_{n_2}] - [x_{n_3}]\| = \|[x_{n_2} - x_{n_3}]\| = \inf_{u \in U} \|x_{n_2} - x_{n_3} + u\|_X < \frac{1}{4},$$

and there exists $y_3 \in U$ such that $\|(x_{n_2} - y_2) - (x_{n_3} - y_3)\|_X < 1/4$. More generally, we have constructed a Cauchy sequence $\{p_k\} := \{x_{n_k} - y_k\} \subset X$. Since X is Banach, $\{p_k\}$ converges to some $p \in X$. Now it remains to notice that

$$\begin{aligned} \|[x_{n_k}] - [p]\| &= \|[x_{n_k} - y_{n_k}] - [p]\| \\ &= \|[p_k] - [p]\| \\ &= \|[p_k - p]\| \to \|0\| = 0. \end{aligned}$$

Hence we've shown $\{[x_{n_k}]\}$ is a convergent subseequence of the Cauchy sequence $\{[x_n]\}$. An $\epsilon/2$ trick would show that $\{[x_n]\}$ converges, and so X/U is Banach.

Problem 6

let X be an infinite dimensional Banach space and let $\{x_n\}_{n\geqslant 1} \subset X$ be any sequence. Let $Y_n := \operatorname{span}(x_1, \dots, x_n)$. Use Baire Category Theorem to show that $\operatorname{span}(x_1, x_2, \dots)$ is not the whole of X. *In other words, no infinite-dimensional Banach space can have a countable Hamel basis.*

Solution

Notice that $Y_{n+1} = Y_n$ if x_{n+1} is not linearly independent from $\{x_1, \dots, x_n\}$. Therefore we may simplify the problem by assuming $\{x_n\}_{n\geqslant 1}$ is a countable Hamel basis of X.

Claim: each Y_n is closed in X. For any n, let $\{v_i\}_{i\geqslant 1}$ be Cauchy where $v_i = \sum_{k=1}^{n} \alpha_k^{(i)} x_k$ with $\alpha_i \in \mathbb{K}$, i.e., a sequence of $\mathbb{K}$–combinations of $(x_1, \dots, x_n)$. It follows that each $\{\alpha_k^{(i)}\}$ must be Cauchy and in particular, since $\mathbb{K}$ is complete, convergent. Hence $\alpha_k := \lim_{i\to\infty} \alpha_k^{(i)}$ is well defined, and $\{v_i\} \to v := \sum_{i=1}^{n} \alpha_k x_k$. Clearly $v \in Y_n$ so Y_n is closed for each n.

Therefore, $\operatorname{span}(x_1, x_2, \dots) = \bigcup_{n\geqslant 1} \operatorname{span}(x_1, \dots, x_n)$, a countable union of closed sets. If this happens to be the whole of X, then by Baire Category Theorem there exists Y_n that is somewhere dense. Hence there exists $y \in Y_n$ such that $B_X(y, r) \subset Y_n$ for some $r > 0$. Notice that, for any other $x \in X$,

$$y + \frac{r}{2} \cdot \frac{x - y}{\|x - y\|} \in B_X(y, r) \subset Y_n.$$

Clearly Y_n is closed under addition since $\mathbb{K}^n$ is. Now we derive a contradiction: notice that

$$x = \frac{2\|x-y\|}{r} \underbrace{\left(y + \frac{r(x-y)}{2\|x-y\|} \right)}_{\in Y_n} + \left(1 - \frac{2\|x-y\|}{r} \right) \underbrace{y}_{\in Y_n} \implies x \in Y_n,$$

so $X \subset Y_n$, but this means X is finite-dimensional, contradiction. Hence no countable Hamel basis exists.

Problem 7

Find a sequence of Lipschitz continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $\sup_n |f_n(x)| < \infty$ for all $x \neq 0$ but that the corresponding Lipschitz constants are unbounded, i.e.,

$$\sup_n \sup_{x \neq y} \frac{|f_n(y) - f_n(x)|}{|y - x|} = \infty.$$

Why does this not contradict the Principle of Uniform Boundedness? In other words, what assumption on the f_n's would one need to add to make sure, using PUB, that such sequence does not exist?

Solution

Consider $f_n(x) := \sin(nx)$. Clearly $f_n(x)$ is n–Lipschitz since $n\cos(nx) \leqslant n$ for all x. Hence the pointwise boundedness is satisfied. But as $n \to \infty$ the Lipschitz constant tends to ∞ as well, hence unbounded. The mistake here is that sine functions are not linear. To make use of PUB, we first need to make sure $\{f_n\} \subset B(\mathbb{R}, \mathbb{R}) \subset L(\mathbb{R}, \mathbb{R})$.

Problem 8

Let X be a Banach space and Y a normed space.

(1) Let $T \in B(X,Y)$. Show that, for every $x \in X$ and $r > 0$,

$$r\|T\| \leqslant \sup_{x' \in B(x,r)} \|T(x')\|.$$

In the remaining steps we will prove the Principle of Uniform Boundedness without using Baire Category Theorem. Suppose that $S \subset B(X,Y)$ is such that

$$\sup_{T \in S} < \infty \text{ for every } x \in X$$

and that $T_n \in S$ satisfies $\|T_n\| \geqslant 4^n$. Set $x_0 := 0$.

(2) Use (1) to deduce that, for every $n \geqslant 1$ there exists $x_n \in X$ such that

$$\|x_n - x_{n-1}\| \leqslant 3^{-n} \text{ and } \|T_n x_n\| \geqslant \frac{2}{3} 3^{-n} \|T_n\|.$$

(3) Deduce that $x_n \to x$ as $n \to \infty$ for some $x \in X$, and that $\|x - xn\| \leqslant 3^{-n}/2$ for all $n \geqslant 0$.

(4) Deduce from (2) and (3) that $\|T_n(x)\| \geqslant 3^{-n}\|T_n\|/6$ for each $n \geqslant 0$, and obtain a contradiction.

Solution: part 1

Let $x \in X$ and $r > 0$ be given. For any $x' \in B(0,r)$, we have

$$\begin{aligned}\|T(x')\| &= \frac{1}{2}\|T[(x+x') + (-x+x')]\| \\ &\leqslant \frac{1}{2}\|T(x+x')\| + \frac{1}{2}\|T(x-x')\| \\ &\leqslant \max\{\|T(x+x')\|, \|T(x-x')\|\}.\end{aligned}$$

Now it remains to take supremum on both sides. The LHS gives

$$\sup_{\|x'\|<r} \|T(x')\| \leqslant \|T\| \sup_{\|x'\|<r} \|x'\| = r\|T\| \text{ since } \|T\| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|} \geqslant \sup_{\|x'\|<r} \frac{\|T(x)\|}{\|x\|}$$

whereas the RHS is nothing but $\sup_{x' \in B(x,r)} \|T(x')\|$. Hence the inequality holds.

Solution: part 2

When $n = 1$, if we define $r_1 := 1/3$ then by (1)

$$\frac{1}{3}\|T_1\| \leqslant \sup_{x_1 \in B(x_0, 1/3)} \|T_1(x_1)\|$$

so there exists $x_1 \in B(x_0, 1/3)$ such that

$$\|x_1 - x_0\| < \frac{1}{3} \text{ and } \frac{2}{3} \cdot \frac{1}{3}\|T_1\| < \frac{1}{3}\|T_1\| \leqslant \|T_1(x_1)\|.$$

Now, for $n = 2$, defining $r_2 := 1/3^2$ gives

$$\frac{1}{9}\|T_2\| \leqslant \sup_{x_2 \in B(x_1, 1/3^2)} \|T_2(x_2)\|$$

and so there exists $x_2 \in B(x_1, 1/3^2)$ such that

$$\|x_2 - x_1\| < \frac{1}{3^2} \text{ and } \frac{2}{3} \cdot \frac{1}{3^2}\|T_2\| < \frac{1}{3^2}\|T_2\| \leqslant \|T_2(x_2)\|.$$

Inductively, we are able to find $x_n \in X$ such that

$$\|x_n - x_{n-1}\| < \frac{1}{3^n} \text{ and } \frac{2}{3} \cdot \frac{1}{3^n}\|T_n\| \leqslant \|T_n(x_n)\|$$

which completes the proof of (2).

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(3) is a one-liner using triangle inequality:
X is Banach, so $\{x_n\}$ is Cauchy $\implies$ $x_n \to x$ for some $x \in X$, and $\|x - x_n\| \leqslant \sum_{i=n}^{\infty} \|x_{n+1} - x_n\| = \sum_{i=n}^{\infty} \frac{1}{3^{n+1}} = \frac{1}{2 \cdot 3^n}$.

For (4), we need (2) and (3) and, once again, triangle inequality:

$$\begin{aligned}
\|T_n(x)\| &= \|T_n(x_n - (x_n - x)\| \\
&= \|T_n(x_n) - T_n(x_n - x)\| \\
&\geqslant \|T_n(x_n)\| - \|T_n(x_n - x)\| && (\Delta \text{ inequality}) \\
&\geqslant \|T_n(x_n)\| - \|T_n\|\|x_n - x\| && (\text{properties of norm}) \\
&\geqslant \frac{2}{3} \cdot \frac{1}{3^n}\|T_n\| - \frac{1}{2} \cdot \frac{1}{3^n}\|T_n\| && (\text{by (2) and (3)}) \\
&= \frac{1}{3^n} \cdot \frac{\|T_n\|}{6} \\
&= \frac{1}{6} \cdot \left(\frac{4}{3}\right)^n \to \infty \text{ as } n \to \infty,
\end{aligned}$$

which contradicts $\sup_{T \in S} \|T(x)\| < \infty$. Hence $\sup_{T \in S} \|T\| < \infty$ and we've proven PUB without using Baire Category Theorem.