MATH 580 Problem Set 3

Qilin Ye

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Problem 1

Let $\{a_n\}_{n\geqslant 1} \subset (0,\infty)$ be such that $\sum_{n\geqslant 1} a_n < \infty$. In this exercise we will show that there exists an unbounded sequence $\{y_n\}_{n\geqslant 1} \subset (0,\infty)$ such that $\sum_{n\geqslant 1} a_n y_n < \infty$.

Consider the map $T : \ell^\infty \to \ell^1$ defined by $T(x_1, x_2, \dots) \coloneqq (x_1 a_1, x_2 a_2, \dots)$.

(1) Show that T is a continuous injection.

(2) Assuming such $\{y_n\}$ does not exist, deduce using IMT that T must be an isomorphism.

(3) Show that any isomorphism between normed spaces X and Y preserves separability.

(4) Use (3) to obtain a contradiction.

Solution

(1) It is clear that if $(x_1, x_2, \dots) \neq (y_1, y_2, \dots)$ then $x_i \neq y_i$ for some $i \geqslant 1$. Then it follows that $x_i a_i \neq y_i a_i$ so $T(x_1, x_2, \dots) \neq T(y_1, y_2, \dots)$. Hence T is injective.

On the other hand, suppose a set of sequences $\{x^{(n)}\}$ with $x^{(n)} \coloneqq (x_1^{(n)}, x_2^{(n)}, \dots)$ converges to $x \coloneqq (x_1, x_2, \dots)$ with respect to ℓ^∞ norm. It follows that $\|x - x^{(n)}\|_{\ell^\infty} = \sup_{i\geqslant 1} |x_i - x_i^{(n)}| \to 0$. On the other hand we have

$$\begin{aligned}
\|T(x) - T(x^{(n)})\|_{\ell^1} &= \sum_{i\geqslant 1} |x_i a_i - x_i^{(n)} a_i| \\
&= \sum_{i\geqslant 1} |x_i - x_i^{(n)}| a_i \\
&\leqslant \sum_{i\geqslant 1} \sup_{i\geqslant 1} |x_i - x_i^{(n)}| a_i \\
&= \|x - x^{(n)}\|_{\ell^\infty} \sum_{i\geqslant 1} a_n
\end{aligned}$$

where the first term $\|\cdot\|_{\ell^\infty} \to 0$ and the second is finite. Hence $T(x^{(n)})$ converges to $T(x)$ in $\|\cdot\|_{\ell^1}$ and this shows the continuity of T.

(2) If such $\{y_n\}$ doesn't exist, then $\sum_{i\geqslant 1} a_n y_n < \infty \implies \{y_n\}$ is bounded, i.e., $\in \ell^\infty$. Notice that, for any sequence $(z_1, z_2, \dots) \in \ell^1$ we have

$$T\left(\frac{z_1}{a_1}, \frac{z_2}{a_2}, \dots\right) = (z_1, z_2, \dots),$$

so under such assumption T is also surjective and hence bijective. Since ℓ^∞ and ℓ^1 are both Banach, by IMT we conclude that T is invertible and thus an isomorphism.

(3) Assume X is separable and let T be the isomorphism. Then there exists a countable subset $E := \{e_i\}_{i\geqslant 1} \subset X$ such that $\overline{\operatorname{span}(E)} = X$. In particular this means that, given any $x_0 \in X$, there exists a sequence of points, all of which are linear combinations of elements of E, that converges to x_0, i.e., $\left\| x_0 - \sum_{i\geqslant 1} \alpha_i^{(n)} e_i \right\| \to 0$ as $n \to \infty$ with all α's $\in \mathbb{K}$. Pick any $y \in Y$. Since T is an isomorphism there indeed exists some $\in X$ with $T(x) = y$, and we may apply the result above. By the continuity of T we also know that $T\Big(\sum_{i\geqslant 1} \alpha_i^{(n)} e_i\Big) \to T(x) = y$ as $n \to 0$. Notice that

$$T\Big(\sum_{i\geqslant 1} \alpha_i^{(n)} e_i\Big) = \sum_{i\geqslant 1} \alpha_i^{(n)} T(e_i) \in \operatorname{span}\{T(e_i)\}.$$

Therefore we've just shown that any $y \in Y$ can be approximated by $\overline{\operatorname{span}\{T(e_i)\}}$. Clearly this set is countable, and so Y is separable as is X.

The other direction Y separable $\implies X$ separable is analogous since the inverse of an isomorphism is also continuous.

(4) From what we've done before we know ℓ^∞ is not separable while ℓ^1 is. Therefore $T : \ell^\infty \to \ell^1$ cannot be an isomorphism, but this cannot happen unless unbounded sequences $\{y_n\}$ exist. In fact, consider

$$\begin{cases} \{a_n\} := (1, 2^{-3}, 3^{-3}, \dots) \in \ell^1 \\ \{y_n\} := (1, 2, 3, \dots) \notin \ell^\infty \end{cases} \implies T(\{y_n\}) = (1, 2^{-2}, 3^{-2}, \dots) \in \ell^1.$$

Problem 2

Show that if $\|\cdot\|$ is a norm induced by an inner product then

$$\|z - x\|^2 + \|z - y\|^2 = \frac{1}{2}\|x - y\|^2 + 2\|z - (x + y)/2\|^2.$$

Solution

Multiplying the LHS by 2, we get

$$2(\|z-x\|^2+\|z-y\|^2) = 2\left(\left\|\left(z-\frac{x+y}{2}\right)-\frac{x-y}{2}\right\|^2+\left\|\left(z-\frac{x+y}{2}\right)+\frac{x-y}{2}\right\|^2\right)$$
$$= 4\left(\left\|z-\frac{x+y}{2}\right\|^2+\left\|\frac{x-y}{2}\right\|^2\right).$$

Factoring out the 2 in the denominator and dividing both sides by 2 give the desired equality.

Problem 3

Consider $f(x) := x$ defined on $[-\pi,\pi]$.

(1) Find the Fourier coefficients in the expansion

$$f(x) = \frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty} c_k e^{ikx}.$$

(2) Use the Parseval identity to show that

$$\sum_{k\geqslant 1}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

Solution

(1) c_0 is simply 0 and c_k for $k \neq 0$ is $1/2\pi$ times the conjugate integral:

$$\begin{aligned}
c_k &= \int_{-\pi}^{\pi} xe^{-ikx}\,\mathrm{d}x \\
&= \left[-\frac{xe^{-ikx}}{ik}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{e^{ikx}}{ik}\,\mathrm{d}x \\
&= \left[-\frac{xe^{-ikx}}{ik}\right]_{-\pi}^{\pi} + \left[-\frac{e^{ikx}}{k^2}\right]_{-\pi}^{\pi} \\
&= \frac{-\pi e^{ik\pi}-\pi e^{-ik\pi}}{ik} - \frac{e^{ik\pi}-e^{-ik\pi}}{k^2} \\
&= -\frac{2\pi\cos k\pi}{ik} - \frac{2i\sin k\pi}{k^2} \\
&= \frac{2k\pi\cos k\pi - 2\sin k\pi}{k^2} i \\
&= \frac{2k\pi\cos k\pi}{k^2} i && (\text{since } \sin k\pi = 0 \text{ for } k\in\mathbb{Z}) \\
&= \frac{2i(-1)^k}{k}.
\end{aligned}$$

(2) Recall that $\{e^{ikx}/\sqrt{2\pi}\}$ is an orthonormal Schauder basis for $L^2(-\pi,\pi)$. Applying Parseval's inequality

to the Fourier expansion derived in (1) gives

$$\int_{-\pi}^{\pi} x^2 \, \mathrm{d}x = \sum_{k=-\infty}^{\infty} \left| \frac{2i(-1)^k}{k\sqrt{2\pi}} \right|^2.$$

The LHS evaluates to $2\pi^3/3$, and the RHS is $2 \cdot \sum_{k=1}^{\infty} \frac{2}{k^2 \pi} = \frac{4}{\pi} \cdot \sum_{k=1}^{\infty} \frac{1}{k^2}$. Therefore

$$\sum_{k \geqslant 1} \frac{1}{k^2} = \frac{2\pi^3}{3} \cdot \frac{1}{4\pi} = \frac{\pi^2}{6}, \text{ as desired.}$$

Problem 4

Let V be an inner product space and let $\{e_i\}_{i \geqslant 1}$ be an orthonormal sequence in V.

(1) Use Bessel's inequality to show that for any $x \in V$,

$$\#\{i : |(x, e_i)| > M\} \leqslant \frac{\|x\|^2}{M^2}.$$

(2) Let E be an uncountable orthonormal set in V. Use (1) to show that then for each $x \in V$ the set $\{e \in E : (x, e) \neq 0\}$ is at most countable.

Now let H be a Hilbert space and $\{e_i\}_{i \geqslant 1}$ an orthonormal Schauder basis.

(3) Show that

$$(u, v) = \sum_{i \geqslant 1} (u, e_i)(e_i, v)$$

for every $u, v \in H$ (*i.e., the inner product then equals the* ℓ^2 *inner product of the Fourier coefficients*).

(4) Show that the *Hilbert cube*

$$Q := \left\{ \sum_{i=1}^{\infty} \alpha_i e_i : |\alpha_i| \leqslant \frac{1}{i} \right\}$$

is a compact subset of H.

Solution

(1) Let $I := \{i \in \mathbb{N} : |(x, e_i)| > M\}$ be an index set. Suppose for contradiction that $\#\{i : |(x, e_i)| > M\}$ is at

least $\left\lfloor \frac{\|x\|^2}{M^2} \right\rfloor + 1$ where $\lfloor \cdot \rfloor$ denotes the *floor function*. Then,

$$\begin{aligned}\sum_{i\geqslant 1} |(x, e_i)|^2 &= \sum_{i\in I} |(x, e_i)|^2 + \sum_{i\notin I} |(x, e_i)|^2 \\ &> M^2 \left(\left\lfloor \frac{\|x\|^2}{M^2} \right\rfloor + 1 \right) + 0 \\ &> M^2 \cdot \frac{\|x\|^2}{M^2} = \|x\|^2,\end{aligned}$$

contradicting Bessel's inequality. Therefore the original inequality must be true.

(2) (Attempt 1) Since

$$\{e \in E : (x, e) \neq 0\} = \bigcup_{i\geqslant 1} \{e \in E : |(x, e)| > \frac{1}{i}\},$$

our target set is therefore a countable union of countable sets and is therefore countable.

(Attempt 2) Let E be an uncountable basis of V. Then for any $x \in V$ we have $x = \sum_{i\in I} (x, e_i) e_i$ where I is some index set and each $e_i \in E$. Applying Parseval's identity gives

$$\|x\|^2 = \sum_{i\in I} |(x, e_i)|^2,$$

and it again becomes clear that, for each $M > 0$, there can only be finitely many e_i's with $|(x, e_i)| > M$. Taking $M := 1, 1/2, 1/3, \ldots$ implies that there are at most countably many e_i's whose inner product with x is nonzero.

(3) This is because

$$\begin{aligned}\sum_{i\geqslant 1} (u, e_i)(e_i, v) &= \sum_{i\geqslant 1} ((u, e_i) e_i, v) \\ &= \left(\sum_{i\geqslant 1} (u, e_i) e_i, v \right) \\ &= (u, v)\end{aligned}$$

since $\{e_i\}_{i\geqslant 1}$ is a Schauder basis.

Problem 5

In this exercise we will show that there exists $f \in X := g \in C([\pi, \pi]; \mathbb{R}) : g(-\pi) = g(\pi)$ (equipped with the $\|\cdot\|_{\max}$ norm) whose Fourier series does not converge at $x = 0$. Let f_n denote the n^{th} Fourier sum of f,

$$f_n(x) := \frac{1}{2\pi} \sum_{k=-n}^{n} \left(\int_{-\pi}^{\pi} f(t) e^{ikt} \, \mathrm{d}t \right) e^{-ikx}$$

and set $S_n f := f_n(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_n(t)\,\mathrm{d}t$ where $K_n(t) := \sum_{k=-n}^{n} e^{ikt}$.

(1) Show that $S_n \in B(X,\mathbb{R})$ for every n with $\|S_n\| = \frac{1}{2\pi}\int_{-\pi}^{\pi}|K_n(t)|\,\mathrm{d}t$. *It's sufficient to point out why the proof of Ex. 3.1.1 could be used here.*

(2) Show that

$$K_n(t) = \frac{\sin(n+1/2)t}{\sin t/2} \text{ for } t \neq 0$$

and $K_n(0) = 2n+1$.

(3) Show that $\|S_n\| \to \infty$ as $n \to \infty$.

(4) Deduce from (a) and (c) that there exists $f \in X$ such that $|f_n(0)| \to \infty$ as $n \to \infty$. *This proves the claim.*

(5) Explain why the existence of such f does not contradict Corollary 6.3.2.

Solution

(1) The claim follows from the fact that $K_n(t) \in C([-\pi,\pi])$.

(2) Notice that $K_0(t) = \sum_{k=-n}^{n} 1 = 2n+1$. Otherwise we can transform this sum into a geometric series:

$$\begin{aligned}
\sum_{k=-n}^{n} e^{ikt} &= e^{i(-k)t} + \cdots + e^{-it} + 1 + e^{it} + \cdots + e^{ikt} \\
&= \sum_{j=0}^{2n} e^{-ikt}(e^{it})^j \\
&= \frac{e^{i(n+1)t} - e^{-int}}{e^{it} - 1} \\
&= \frac{e^{i(n+0.5)t} - e^{i(n+0.5)t}}{e^{i(0.5)t} - e^{i(-0.5)t}} \\
&= \frac{2i\sin[(n+1/2)t]}{2i\sin(t/2)}
\end{aligned}$$

(3) Here we use the fact that $\int |\cdot|$ is a monotone increasing function so that it's not larger than its right

endpoint Riemann sum:

$$\begin{aligned}
\|S_n\| &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin[(n+1/2)t]}{\sin(t/2)}\right|\,\mathrm{d}t \\
&\sim \int_{-\pi}^{\pi}\left|\frac{\sin[(n+1/2)t]}{\sin(t/2)}\right|\,\mathrm{d}t && (\sim\text{: ignoring the constant}) \\
&\sim \int_{0}^{\pi}\left|\frac{\sin[(n+1/2)t]}{\sin(t/2)}\right|\,\mathrm{d}t && (\text{even function}) \\
&\geqslant \int_{0}^{\pi}\left|\frac{\sin[(n+1/2)t]}{t/2}\right|\,\mathrm{d}t && (|\sin x|\leqslant|x|) \\
&= \sum_{j=1}^{2n}\int_{\frac{(j-1)\pi}{2n+1}}^{\frac{j\pi}{2n+1}}|\sim|\,\mathrm{d}t \\
&\geqslant \sum_{j=1}^{2n}\left|\frac{2n+1}{j\pi}\right|\int_{\frac{(j-1)\pi}{2n+1}}^{\frac{j\pi}{2n+1}}\left|\sin[(x+0.5)t]\right|\,\mathrm{d}t && (\text{taking the max denominator for all}) \\
&= \sum_{j=1}^{2n}\left|\frac{2n+1}{j\pi}\right|\cdot\frac{1}{n+0.5}\cdot\underbrace{\int_{(j-1)\pi/2}^{j\pi/2}|\sin\tilde{t}|\,\mathrm{d}\tilde{t}}_{=1} && (\text{simple chain rule with } \tilde{t}:=(x+0.5)t) \\
&= \sum_{j=1}^{2n}\frac{2n+1}{j\pi(n+0.5)} \\
&\sim \sum_{j=1}^{2n}\frac{1}{j}\to\infty \text{ as } n\to\infty.
\end{aligned}$$

(4) Recall that $C([-\pi,\pi];\mathbb{R})\subset C_b(\mathbb{R};\mathbb{R})$ is Banach. Suppose the opposite of the claim is true. Then because $\sup|f_n(0)|<\infty$ we know that $\|S_n\|$ is also bounded, contradicting (3).

(5) Because Corollary 6.3.2 doesn't require the approximating family of functions to be uniformly bounded.

Problem 6

Use the WAT to show that any $f\in C([0,1];\mathbb{R})$ can be approximated (in $\|\cdot\|_{\max}$) by a cosine polynomial, i.e., an element of

$$\mathcal{C}:=\left\{\sum_{k=0}^{n}a_k\cos(k\pi x) : n\in\mathbb{N}, a_k\in\mathbb{R}\right\}.$$

Problem 7

Let $X:=C([-1,1])$ equipped with the supremum norm and let

$$U:=\left\{g\in X : \int_{-1}^{0}g=\int_{0}^{1}g=0\right\}$$

a closed subspace of X. Let $f \in X$ be any function with $\int_{-1}^{0} f = 1$ and $\int_{0}^{1} f = -1$.

(1) Show that $\operatorname{dist}(f, U) = 1$ but $\|f - g\| > 1$ for all $g \in U$, i.e., *there is no closest point to f in U.*

(2) Explain why the previous part does not contradict Lemma 6.9.

Solution

(1) Let $\{g_n\}$ be a sequence of functions such that

$$f(x) - g_n(x) = \begin{cases} \frac{2n}{2n-1} & x \in [-1, -\frac{1}{n}] \\ -\frac{2n^2 x}{2n-1} & x \in (-\frac{1}{n}, \frac{1}{n}) \\ -\frac{2n}{2n-1} & x \in [\frac{1}{n}, 1] \end{cases} .$$

One can easily check that $f(x) - g_n(x)$ is piecewise linear and also continuous, so $g_n(x) \in C([-1, 1])$. It is obtained by connecting $(-1, -1), (-\frac{1}{n}, \frac{2n}{2n-1}), (\frac{1}{n}, -\frac{2n}{2n-1})$, and $(1, -1)$. Notice that

$$\int_{-1}^{0} g = \int_{-1}^{0} f - \int_{-1}^{0} (f - g) = 1 - 1 = 0$$

and likewise $\int_{0}^{1} g = 0$. Therefore $\{g_n\} \subset U$. Since $\|g_n - f\| = (2n-1)/2n$, letting $n \to \infty$ gives $\sup \|g_n - f\| = 1$, which suggests that $\operatorname{dist}(f, U) \leqslant 1$.

Now we first show $\|f - g\| > 1$ for all $g \in U$. Suppose there exists some $g \in U$ such that $\|f - g\| \leqslant 1$. If it's strictly < 1 then

$$\int_{-1}^{0} f - g \leqslant \int_{0}^{1} |f - g| \leqslant \|f - g\|(1 - 0) < 1,$$

which means $\int_{-1}^{0} g > 0$, contradiction. On the other hand, similar tot he second $\leqslant$ above, if $\|f - g\| = 1$, to make sure $\int_{-1}^{0} g = 0$ it must be the case that $f(x) - g(x) = 1$ for all $x \in [-1, 0]$. Likewise, $g(x) - f(x) = 1$ for all $x \in [0, 1]$. Then we have $g(0) - f(0) = f(0) - g(0) = 1$, clearly a contradiction. This proves both claims of part 1 (i.e., distance = 1 and no closest point exists).

(2) In $C([-1, 1])$, the supremum norm is *not* induced by the inner product, hence $(C([-1, 1]), \|\cdot\|_{\sup})$ is not Hilbert and the Lemma does not apply. Simply consider $f := x - 1$ and $g := x + 2$ so that $\|f + g\|^2 + \|f - g\|^2 = 3^2 + 3^2 = 18$ whereas $2(\|f\|^2 + \|g\|)^2 = 2(2^2 + 3^2) = 26$ which fails to satisfy the parallelogram law.