MATH 580 Problem Set 4

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Problem 1

A Banach space is called *uniformly convex* if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x - y\| > \epsilon, \|x\| = \|y\| = 1 \implies \left|\frac{x+y}{2}\right| < 1 - \delta.$$

(a) Show that every Hilbert space is uniformly convex.

Solution

Applying the parallelogram law to x and y with $\|x\| = \|y\| = 1$ gives

$$\begin{aligned}\epsilon^2 < \|x-y\|^2 &= 2(\|x\|^2 + \|y\|^2) - \|x+y\|^2 \\ &= 4 - 4\|(x+y)/2\|^2.\end{aligned}$$

Therefore $\|(x+y)/2\|^2 < 1 - \epsilon^2/4$, and setting $\delta := 1 - \sqrt{1-\epsilon^2/4}$ finishes the proof.

(b) Use the following *Clarkson's inequalities*

$$\|(f+g)/2\|_{L^p}^p + \|(f-g)/2\|_{L^p}^p \leqslant \frac{1}{2}(\|f\|_{L^p}^p + \|g\|_{L^p}^p) \qquad \text{for } p \in [2,\infty)$$

$$\|(f+g)/2\|_{L^p}^q + \|(f-g)/2\|_{L^p}^q \leqslant \left(\frac{1}{2}\|f\|_{L^p}^p + \frac{1}{2}\|g\|_{L^p}^p\right)^{q/p} \qquad \text{for } p \in (1,2]$$

(where $1/p + 1/q = 1$) to show that $L^p(\Omega)$ is uniformly convex for $p \in (1,\infty)$ for any Ω.

Solution

Suppose $\|f\| = \|g\| = 1$ and $\|f - g\| > \epsilon$ (all with respect to $\|\cdot\|_{L^p}$). If $p \in [2, \infty)$, the first inequality gives

$$\begin{aligned}\|(f+g)/2\|_{L^p}^p &\leqslant 1 - \|(f-g)/2\|_{L^p}^p \\ &< 1 - (\epsilon/2)^p,\end{aligned}$$

so taking $\delta := 1 - (1 - (\epsilon/2)^p)^{-p}$ proves the uniform convexity. On the other hand, if $p \in (1, 2]$, then

$$\begin{aligned}\|(f+g)/2\|_{L^p}^q &\leqslant 1 - \|(f-g)/2\|_{L^p}^q \\ &< 1 - (\epsilon/2)^q\end{aligned}$$

so, similarly, taking $\delta := (1 - (1 - \epsilon/2)^q)^{-q}$ completes the proof.

(c) Show that any uniformly convex Banach space X is strictly convex, i.e., if $x, y \in X$, $x \neq y$, and $\|x\| = \|y\| = 1$ then $\|x + y\| < 2$.

Solution

This follows immediately from the definition. If $\|x\| = \|y\| = 1$ but $x \neq y$ then $\|x - y\| > 0$, and so there exists $\delta > 0$ such that $\|(x - y)/2\| < 1 - \delta < 1$, i.e., $\|x - y\| < 2 - 2\delta < 2$.

(d) In this part we will generalize the "closest point lemma" (Lemma 6.9) to the case of uniformly convex Banach spaces. *In particular, by (1), this gives another proof of Lemma 6.9.* Let A be a closed and convex subset of a uniformly Banach space X and let $X \setminus A$. Let $\{a_n\}_{n \geqslant 1} \subset A$ be such that $\|x - a_n\| \to \inf_{a \in A} \|x - a\| =: d$.

(d1) Set $x_n := (x - a_n)/\|x - a_n\|$ and use the convexity of A to show that

$$\|(x_n + x_m)/2\| \geqslant \frac{d}{2}\left(\frac{1}{\|x - a_n\|} + \frac{1}{\|x - a_m\|}\right).$$

Hint: try writing $x_n + x_m = (\|x - a_n\|^{-1} + \|x - a_m\|^{-1})(c_1(x - a_n) + c_2(x - a_m))$ *for some* $c_1, c_2 \in \mathbb{R}$.

Solution

By the hint, we may define $c_1 := \dfrac{\|x - a_n\|^{-1}}{\|x - a_n\|^{-1} + \|x - a_m\|^{-1}}$ and $c_2 := \dfrac{\|x - a_m\|^{-1}}{\|x - a_n\|^{-1} + \|x - a_m\|^{-1}}$. Then

$$\begin{aligned}\text{RHS} &= \frac{x - a_n}{\|x - a_n\|} + \frac{x - a_m}{\|x - a_m\|} \\ &= x_m + x_n.\end{aligned}$$

Notice that $c_1 + c_2 = 1$. Therefore

$$\begin{aligned}
\|(x_n + x_m)/2\| &= \frac{1}{2}\left(\frac{1}{\|x - a_n\|} + \frac{1}{\|x - a_m\|}\right)((c_1 + c_2)x - (c_1 a_n + c_2 a_m)) \\
&= \frac{1}{2}\left(\frac{1}{\|x - a_n\|} + \frac{1}{\|x - a_m\|}\right)(x - \underbrace{(c_1 a_n + c_2 a_m)}_{\in A \text{ by convexity}}) \\
&\geqslant \frac{d}{2}\left(\frac{1}{\|x - a_n\|} + \frac{1}{\|x - a_m\|}\right).
\end{aligned}$$

(d2) Deduce that $\|(x_n + x_m)/2\| \to 1$ as $\min(m, n) \to \infty$.

Solution

By (d1), $\frac{d}{2}(\|x - a_n\|^{-1} + \|x - a_m\|^{-1}) \leqslant \|(x_n + x_m)/2\|$, and by (c), since $\|x_n\| = \|x_m\|$ we also have $\|(x_n + x_m)/2\| < 1$. Since

$$\lim_{n,m\to\infty} \frac{d}{2}\left(\frac{1}{\|x - a_n\|} + \frac{1}{\|x - a_m\|}\right) = \frac{d}{2}\left(\frac{1}{d} + \frac{1}{d}\right) = 1,$$

we conclude that the same thing happens for $\|(x_n + x_m)/2\|$.

(d3) Use uniform convexity of X to show that $\{x_n\}$ is Cauchy.

Solution

Suppose not, then there exists $\epsilon > 0$ such that, for all $N \in \mathbb{N}$, there exists $m > n > N$ with $\|x_m - x_n\| > \epsilon$. Taking $N \to \infty$ contradicts (d2) since in this case $\min(m, n) \to \infty$ as well. Hence $\{x_n\}$ is Cauchy.

(d4) Deduce that $\{a_n\}$ is Cauchy as well.

Solution

$$\begin{aligned}
\|a_n - a_m\| &= \|(x-a_n) - (x-a_m)\| \\
&= \left\| (x-a_n) - \frac{\|x-a_n\|}{\|x-a_m\|}(x-a_m) + \frac{\|x-a_n\| - \|x-a_m\|}{\|x-a_m\|}\|x-a_n\| \right\| \\
&\leqslant \left\| (x-a_n) - \frac{\|x-a_n\|}{\|x-a_m\|}(x-a_m) \right\| + \left| \frac{\|x-a_n\| - \|x-a_m\|}{\|x-a_m\|} \right| \|x-a_m\| \\
&= \frac{1}{\|x-a_n\|}\|x_n - x_m\| + \dots \\
&\to 0 + 0 = 0
\end{aligned}$$

where the first 0 is given by the Cauchy-ness of $\{x_n\}$ and the second by the construction $\lim \|x-a_n\| = \lim \|x-a_m\| = d$. Hence $\{a_n\}$ is Cauchy too.

(d5) Deduce the existence of $\hat{a} \in A$ such that $\|x - \hat{a}\| = d$.

Solution

Since X is Banach, $\{a_n\}$ converges to some $\hat{a} \in X$. Then because A is closed, $\hat{a} \in A$. Such $\hat{a}$ is precisely the one we are looking for.

(e) Suppose that X is a strictly convex Banach space. Let $A \subset X$ be closed and convex and $x \in X \setminus A$. Show that if there exists a closest point $\hat{a} \in A$ to x then it's unique. *In other words, the closest point $\hat{a} \in A$ that we have found in (d5) is unique even in strictly convex Banach spaces.*

Solution

Suppose, for contradiction, that there exists a different $a' \in A$ with $\|x-\hat{a}\| = \|x-a'\|$. By strict convexity of A, we know that

$$\left\| \frac{x-\hat{a}}{\|x-\hat{a}\|} + \frac{x-a'}{\|x-a'\|} \right\| = \frac{\|2x - \hat{a} - a'\|}{d} < 2.$$

Therefore $\|x - (\hat{a}+a')/2\| < d$. But this gives a contradiction as $\hat{a}$ is assumed to be a closest point.

(f) Deduce that $L^\infty(0,1)$ is not uniformly convex.

Solution

Simply take $f, g \in L^\infty(0,1)$ with $f \equiv 1$ and $g(x) = x$. It follows that (with $\|\cdot\|_{\text{sup}}$) $\|f\| = \|g\| = 1$ whereas $\|f+g\| = 2$. This means $L^\infty(0,1)$ is not strictly convex, which implies it's not uniformly convex.

Problem 2

Let H be a real Hilbert space and $B : H \times H \to \mathbb{R}$ is such that:

(i) $B(x,y)$ is linear in both x and y,

(ii) $|B(x,y)| \leqslant c\|x\|\|y\|$ for some $c > 0$, for all $x, y \in H$,

(iii) $B(x,y) \geqslant b\|x\|^2$ for some $b > 0$ and all $x \in H$, and

(iv) $B(x,y) = B(y,x)$ for all $x, y \in H$.

In other words, $B(\cdot,\cdot)$ is an inner product on H but not ncessarily the one that induces the norm $\|\cdot\|$.

(a) Show that $u \in H$ minimizes

$$F(u) := \frac{1}{2}B(u,u) - f(u),$$

where $f \in H^*$ is given, if and only if

$$B(u,v) = f(v) \text{ for every } v \in H.$$

(Hint: consider $F(u+tv)$ where $t \in \mathbb{R}$.)

Solution

For $\Longrightarrow$, assume u is the minimizer. Then for all $v \in H$ and all $t \in \mathbb{R}$ we have

$$F(u+tv) - F(u) \geqslant 0. \tag{1}$$

On the other hand,

$$\begin{aligned}
F(u+tv) - F(u) &= \frac{1}{2}[(B(u+tv, u+tv) - B(u,u)] - (f(u+tv) - f(u)) \\
&= \frac{1}{2}[2tB(u,v) + B(tv,tv)] - (u+tv, x) + (u,x) && \begin{cases}\text{for some } x \in H \text{ by} \\ \text{Riesz Representation Thm}\end{cases} \\
&= tB(u,v) + \frac{1}{2}B(tv,tv) - (tv,x) \\
&\leqslant tB(u,v) + \frac{1}{2}ct^2\|v\|^2 - (tv,x) && \begin{cases} c > 0, \text{ by applying (ii), since} \\ B(tv,tv) = |B(tv,tv)| \end{cases} \\
&= \frac{c\|v\|^2 t^2}{2} + (B(u,v) - f(v))t =: h(t).
\end{aligned}$$

Clearly h is concave up and taking derivative gives $h'(t) = c\|v\|^2 t + (B(u,v) - f(v))$, so $h(t)$ attains its minimum at $t_0 = \dfrac{B(u,v) - f(v)}{c\|v\|^2}$. Substituting t_0 back into the equation we see that the minimum is

$$\begin{aligned} h(t_0) &= \frac{\|v\|^2(B(u,v) - f(v))^2}{2c^2\|v\|^4} - \frac{(B(u,v) - f(v))^2}{c\|v\|^2} \\ &= \frac{c(B(u,v) - f(v))^2 - 2(B(u,v) - f(v))^2}{2c\|v\|^2} \\ &= -\frac{(B(u,v) - f(v))^2}{2c\|v\|^2} \leqslant 0. \end{aligned} \tag{2}$$

Therefore should (1) and (2) be both true, it must be the case that

$$-\frac{(B(u,v) - f(v))^2}{2c\|v\|^2} = 0 \text{ for all } v \in H \implies B(u,v) = f(v) \text{ for all } v \in H.$$

For $\Longleftarrow$, suppose $B(u,v) = f(v)$ for all $v \in H$. Then,

$$\begin{aligned} F(u + tv) - F(u) &= \frac{1}{2}[2tB(u,v) + B(tv,tv)] - tf(v) \\ &= tB(u,v) + \frac{1}{2}B(tv,tv) - tB(u,v) \\ &= \frac{1}{2}B(tv,tv) \geqslant 0, \end{aligned}$$

which shows u is a minimizer.

(b) Let $A \subset H$ be a closed, convex (and nonempty) set. Show that F is bounded below on A, and that there exists a unique minimizer of F on A. *Hint: try using Riesz Representation Theorem and the closest point lemma. Hint 2: consider B as the inner product.*

Solution

Boundedness:

$$\begin{aligned} F(u) &= \frac{1}{2}B(u,u) - f(u) \\ &\geqslant b\|u\|^2 - (u,x) && \text{(For some } b > 0\text{, by (iii) and Riesz RT)} \\ &\geqslant b\|u\|^2 - \|u\|\|x\| && \text{(By Cauchy-Schwarz (on the induced norm))} \end{aligned}$$

which is bounded from below by $F(u_0)$ where u_0 satisfies $F'(u_0) = 0$ (since the leading coefficient of this quadratic polynomial is $b > 0$, positive).

Existence *&* uniqueness: we first show that there exists a unique element of H that acts as a minimizer of F on the entire H. Since $B(\cdot,\cdot)$ can be seen as an inner product and $f \in H^*$, there exists $x \in H$ such that $f(v) = B(v,x)$ for all $v \in H$. By (a), $B(u,v) = f(v)$ for every $v \in H$ if and only if u minimizes $F(u)$. However, since $f(v) = B(v,x)$ and $B(u,v) \overset{\text{(iv)}}{=} B(v,u)$, and $B(v,x) = B(v,u)$ for all $v \in H$ if and only if $v = u$, we conclude that

$$u \text{ minimizes } F(u) \iff u = x.$$

Hence the existence and uniqueness of minimizer. Now it remains to show $x \in A$. If this is the case then we are immediately done. Otherwise we claim that the orthogonal projection of x onto A, i.e., the "closest point", is the minimizer of F on A. Let x_0 be such point, i.e., $\|x - x_0\| = \inf_{a \in A} \|u - a\|$. We will show that $F(a) \geqslant F(x_0)$ for all $a \in A$:

$$\begin{aligned}
F(a) - F(x_0) &= \frac{1}{2}[B(a,a) - B(x_0,x_0)] - (f(a) - f(x_0)) \\
&= \frac{1}{2}[B(a,a) - B(x_0,x_0)] - (B(a-x_0,x)) && \text{(Riesz)} \\
&= \frac{1}{2}[B(a,a) - B(x_0,a) + B(x_0,a) - B(x_0,x_0) - B(a-x_0,x) - B(x,a-x_0)] \\
&= \frac{1}{2}[B(a-x_0,a) + B(x_0,a-x_0) - B(a-x_0,x) - B(x,a-x_0)] \\
&= \frac{1}{2}[B(a-x_0,a-x) + B(x_0-x,a-x_0)] \\
&= \frac{1}{2}[B(a-x_0,a-x_0) + B(a-x_0,x_0-x) + B(x_0-x,a-x_0)].
\end{aligned}$$

The first term is nonnegative by definition of $\|\cdot\|$ and positive whenever $a \neq x_0$. The second and third are nonnegative because $(a - x_0, x - x_0) \leqslant 0$ by Lemma 6.9. Furthermore, 0 can only be attained when $a = x_0$. Hence x_0 is indeed the unique minimizer of F on A, done.

Problem 3

Let M be a closed subspace of a Hilbert space H. Show that H/M (recall PS2.5) is isometrically isomorphic to $M^\perp$ viat he mapping $T : H/M \to M^\perp$ defined by $T([x]) := P^\perp x$, where $P^\perp$ is the orthogonal projection onto $M^\perp$.

Solution

The proof roughly divides into four parts:

(1) Injectivity of T. Suppose $T([x]) = T([y])$, i.e., $P^\perp x = P^\perp y$. Then (since M is closed†)

$$P^\perp(x-y) = 0 \implies x - y \in (M^\perp)^\perp \overset{\dagger}{\implies} x - y \in M \implies [x] = [y].$$

(2) Surjectivity of T. For any $m \in M^\perp$, simply notice that $T([m]) = P^\perp m = m$.

(3) $\|T\| = 1$. For any $x \in H$, there exists a decomposition $x = x_1 + x_2$ where $x_1 \in M$ and $x_2 \in M^\perp$. Then,

$$\|T[x]\|_H = \|P^\perp x\|_H = \|x_2\|_H = \|[x]\|_{H/M}$$

where the last equality is because

$$\|[x]\|_{H/M} = \inf_{x' \in M} \|x + x'\| = \inf_{x' \in M} \|x' + x_2\| = \inf_{x' \in M} \sqrt{\|x'\|^2 + \|x_2\|^2} = \|x_2\|_H.$$

(4) $\|T^{-1}\| = 1$. First, its boundedness is guaranteed by IMT since H/M is complete (H is Banach and so is H/M; cf. PS2.5). Then

$$\|[x]\|_{H/M} = \|T^{-1}T([x])\|_{H/M} = \|T[x]\|_H$$

and we conclude that T is indeed an isometry; $H/M \equiv M^\perp$.

Problem 4

Let X be a normed space and U, Y two subspaces of X.

(a) Show that every $x \in X$ can be uniquely decomposed into $x = u + y$, where $u \in U$ and $y \in Y$, if and only if $U + Y = X$ and $U \cap Y = \{0\}$.

Solution

For $\Longrightarrow$, suppose the unique decomposition exists.

(1) $X = U + Y$: clearly $U + Y \subset X$, so it suffices to show $X \subset U + Y$, but this is precisely what the existence of decomposition means, since any $x \in X$ is also in $U + Y$.

(2) $U \cap Y = \{0\}$: suppose not so there exists nonzero $z \in U \cap Y$. Since $z = 0 + z = z + 0$ and $0 \in U \cap Y$,

$$z = \underbrace{z}_{\in U} + \underbrace{0}_{\in Y} = \underbrace{0}_{\in U} + \underbrace{z}_{\in Y},$$

a contradiction to the uniqueness of decomposition. Hence $U \cap Y = \{0\}$.

For $\Longleftarrow$, suppose $U + Y = X$ and $U \cap Y = \{0\}$.

(1) Existence: since $X \subset U + Y$, for any $x \in X$ there exists $u \in U$ and $y \in Y$ such that $x = u + y$.

(2) Uniqueness: suppose $x = u_1 + y_1 = u_2 + y_2$. It follows that $u_1 - u_2 \in U$ and $y_2 - y_1 \in Y$ are equal. Hence $u_1 - u_2 = y_1 - y_2 \in U \cap Y = \{0\}$ and so indeed $u_1 = u_2, y_1 = y_2$. Hence unique.

(b) Now let X be a Banach space. Show that U, Y are complements (i.e. U, Y are closed and every x admits a unique decomposition in (a)) if and only if there exists $T \in B(X)$ such that $T^2 = T$ and $\text{im}(T) = U$. (*This shows Theorem 7.2.1.*)

Solution

For $\implies$, assume U is complemented by Y. By (a), each $x \in X$ can be uniquely decomposed into $x = u + y$ for some $u \in U, y \in Y$. Define a mapping $T : X \to U$ by $x \mapsto u$.

(1) $T \in B(x, y)$: linearity is trivial; it suffices to show boundedness. Since X is not ncessarily Hilbert, the original proof of $\|P_U\| = 1$ breaks down. Neither does sequential continuity work. Hence we resort to the Closed Graph Theorem. Let $\{(x_n, T(x_n))\} \subset G(T)$ be a sequence that converges to $(x, u) \in X \times U$. We want to show that it converges in $G(T)$ (the graph), i.e., $u = T(x)$.

Decompose x_n into $u_n + y_n$. It follows that $T(x_n) = u_n \to u \in X$. Hence $y_n \to x - u \in X$. On the other hand, by assumption, the closures of U and Y suggest that $u \in U$ and $x - u \in Y$. Therefore

$$x = \underbrace{u}_{\in U} + \underbrace{(x - u)}_{\in Y}$$

is a decomposition of x, and by assumption it is *the* decomposition of x. Therefore $T(x) = u$ as desired, and $G(T)$ is closed. By the Closed Graph Theorem this means $T \in B(X)$.

(2) $T^2 = T$: immediate since $T(T(x)) = T(u) = u$.

(3) $\text{im}(T) = U$: $\subset$ by definition of decomposition and of T; $\supset$ because for all $u \in U$ we have $T(u) = u$.

For $\impliedby$, simply take $U := \text{im}(T)$ and $Y := \ker(T)$. We want to show $U + Y = X$ and $U \cap Y = \{0\}$.

(1) $U + Y = X$: the direction $\text{im}(T) + \ker(T) \subset X$ is trivial. To show $X \subset \text{im}(T) + \ker(T)$, we use the idempotency of T: for all $x \in X$,

$$x = (x - T(x)) + T(x)$$

where $T(x) \in \text{im}(T)$ and $x - T(x) \in \ker(T)$ since

$$T(x - T(x)) = T(x) - T^2(x) = 0.$$

(2) $U \cap Y = \{0\}$: since $\ker(\text{id} - T) = \text{im}(T)$, the claim is equivalent to $\ker(\text{id} - T) \cap \ker(T) = \{0\}$. This is indeed true because if $(\text{id} - T)(x) = T(x) = 0$ then $\text{id}(x) = 0 \implies x = 0$.

Problem 5

In this problem we show that the projection theorem, Proposition 6.10, is not true in inner product spaces that are not Hilbert. Let $X := c_{00}$ be equipped with the ℓ^2 norm, and

$$U := \left\{\{x_n\}_{n\geqslant 1} \in c_{00} : \sum_{n\geqslant 1} x_n/n = 0\right\}.$$

(a) Show that there exists a unique $x^* \in X^*$ such that $U = \ker(x^*)$ and deduce that U is a closed linear subspace of c_{00}.

Solution

Intuitively we think of $x^* \in X^*$ defined by $x^* : \{x_n\} \mapsto \sum_{n\geqslant 1} x_n/n$. Indeed x^* is well-defined and linear, and it is bounded because (by Cauchy-Schwarz)

$$\left|\sum_{n\geqslant 1} x_n/n\right| \leqslant \sqrt{\sum_{n\geqslant 1} x_n^2}\sqrt{\sum_{n\geqslant 1}(1/n^2)} = \frac{\pi^2}{6}\|\{x_n\}\|_{\ell^2}.$$

It follows that $U = \ker(x^*)$, and since $x^* \in X^*$, the kernel is indeed closed.
Now it remains to show uniqueness. Suppose there exists some $y^* \in X^*$ with $\ker(x^*) = \ker(y^*)$. By Riesz Representation Theorem, if we define x^* on the entire ℓ^2 (which is then Hilbert), there exists $a = \{a_n\}_{n\geqslant 1} \in \ell^2$ such that $y^*(\{x_n\}_{n\geqslant 1}) = \sum_{n\geqslant 1} a_n x_n$.
Notice that, for all $n \in \mathbb{N}$, the sequence with first term 1, n^{th} term $(-n)$, and 0 everywhere else, is in the kernel of x^*, i.e., $(1, 0, \dots, 0, -n, 0, \dots) \in \ker(x^*)$. By assumption this sequence is also $\in \ker(y^*)$. Therefore $a_1 - na_n = 0 \implies a_n = a_1/n$. Hence

$$y^*(\{x_n\}) = \sum_{n\geqslant 1} \frac{a_1 x_n}{n} = a_1 x^*(\{x_n\}),$$

and so indeed x^* is unique.

(b) Show that there is no $y \in c_{00}$ such that $x^*(x) = \langle x, y\rangle$ for all $x \in X$, i.e., *the claim of the Riesz Representation Theorem is not valid in* X.

Solution

Suppose there exists some $x = \{x_n\}$ that satisfies $\sum_{i\geqslant 1} y_n/n = \sum_{i\geqslant 1} x_n y_n$. Taking $e^{(i)}$ suggests that $y_i = 1/i$. But then $y \notin c_{00}$, contradiction.

(c) Show that $U^\perp = \{0\}$, and deduce that it is not true that every $x \in X$ can be uniquely decomposed as $x = u + v$ where $u \in U$ and $v \in U^\perp$.

Solution

(1) $U^\perp = \{0\}$: take any $x = \{x_n\} \in U^\perp$. We have $\sum_{i\geqslant 1} x_n y_n = 0$ for any $y = \{y_n\} \in U$. Recall the set of sequences with first term 1, n^{th} term $(-n)$, and 0 everywhere else. These sequences are in $\ker(x^*) = U$, so applying the equality above gives $x_n = x_1/n$. The sequence $\{x_1/n\}_{n\geqslant 1}$ is in c_{00} if and only if $\{x_n\} = 0$. Therefore $U^\perp = \{0\}$.

(2) Decomposition not guaranteed: simply consider $x := (1, 0, \dots)$. If there exists a decomposition, since $U^\perp = \{0\}$ we must have $v = 0$ and $u = x$, but $x \notin U$ since $\sum_{n\geqslant 1} x_n/n = 1 \neq 0$.

Problem 6

Let $\alpha = \{\alpha_i\}_{i\geqslant 1} \in \ell^\infty(\mathbb{C})$ and consider $D_\alpha \in B(\ell^2(\mathbb{C}))$ defined by

$$D_\alpha(x_1, x_2, \dots) := (\alpha_1 x_1, \alpha_2 x_2, \dots).$$

Show that:

(a) $\sigma_{\mathrm{p}}(D_\alpha) = \{\alpha_i\}_{i\geqslant 1}$.

Solution

We first show $\{\alpha_i\} \subset \sigma_{\mathrm{p}}(D_\alpha)$. For any α_i, simply consider $e^{(i)}$:

$$(D - \alpha_i I)(e^{(i)}) = \sum_{n\geqslant 1} (\alpha_n - \alpha_i) e_n^{(i)} = 0 \text{ but } e^{(i)} \neq 0.$$

For the other direction, i.e., $\sigma_{\mathrm{p}}(D_\alpha) \subset \{\alpha_i\}$, suppose for contradiction that there exists some $\lambda \notin \{\alpha_i\}$ such that $(D_\alpha - \lambda I)(x) = 0$ for some $x \neq 0$. It follows that

$$(\alpha_1 x_1, \alpha_2 x_2, \dots) = (\lambda x_1, \lambda x_2, \dots).$$

Since $x \neq 0$, there exists some $n \in \mathbb{N}$ such that $\alpha_n x_n = \lambda x_n \implies \lambda = \alpha_n \in \{\alpha_i\}$, contradiction. Therefore we conclude that $\sigma_{\mathrm{p}}(D_\alpha) = \{\alpha_i\}$.

(b) $\sigma(D_\alpha) = \overline{\sigma_{\mathrm{p}}(D_\alpha)}$.

Solution

Recall that $\sigma_{\mathrm{p}}(D_\alpha) \subset \sigma(D_\alpha)$ and that $\sigma(D_\alpha)$ is closed. Therefore it suffices to show $\sigma(D_\alpha) \subset \overline{\sigma_{\mathrm{p}}(D_\alpha)}$, and this is equivalent to showing the other inclusion of the complements, i.e., if $\lambda \notin \overline{\sigma_{\mathrm{p}}(D_\alpha)}$ then $\lambda \notin \sigma(D_\alpha)$, i.e., $\lambda \in S(D_\alpha)$. Fix this λ. We need to show $D_\alpha - \lambda I$ is invertible.

(1) Injectivity: suppose $(D_\alpha - \lambda I)(x) = (D_\alpha - \lambda I)(y)$. Then

$$((\alpha_1 - \lambda)x_1, (\alpha 2 - \lambda)x_2, \dots) = ((\alpha_1 - \lambda)y_1, (\alpha_2 - \lambda)x_2, \dots) \implies (x_1, x_2, \dots) = (y_1, y_2, \dots).$$

(2) Surjectivity: for any $y = \{y_n\} \in \ell^2(\mathbb{C})$, define

$$x := \left(\frac{y_1}{\alpha_1 - \lambda}, \frac{y_2}{\alpha_2 - \lambda}, \dots\right).$$

This is well-defined because $\lambda \notin \overline{\{\alpha_i\}}$ by assumption, which means $\inf_{i \geqslant 1} |\alpha_i - \lambda| = d$ for some $d > 0$. Furthermore, $x \in \ell^2(\mathbb{C})$ because

$$\|x\|_{\ell^2}^2 = \sum_{i \geqslant 1} \frac{|y_i|^2}{|\alpha_i - \lambda|^2} \leqslant \sum_{i \geqslant 1} \frac{|y_i|^2}{d^2} = \frac{\|y\|_{\ell^2}^2}{d^2} < \infty.$$

(3) Boundedness: trivial.

(4) Boundedness of inverse: immediate by IMT since $\ell^\infty(\mathbb{C})$ and $\ell^2(\mathbb{C})$ are both Banach.

(c) For every compact set $K \subset C$ there exists $\alpha \in \ell^\infty(\mathbb{C})$ such that $K = \sigma(D_\alpha)$.

Solution

Since $K \subset C$ and $\mathbb{C}$ is separable, the set $\{a + bi : a, b \in \mathbb{Q} \text{ and } a + bi \in K\}$ forms a countable dense subset of K. Furthermore, since K is compact, it is closed and bounded. Enumerating this set as $\alpha := \{\alpha_i\}_{i \geqslant 1}$ we have therefore obtained a sequence in $\ell^\infty(\mathbb{C})$. By (a), $\sigma_{\mathrm{p}}(D_\alpha) = \{\alpha_i\}$, but since α is dense in K, $\overline{\sigma_{\mathrm{p}}(D_\alpha)} = \overline{\{\alpha_i\}} = K$, and by (b), $K = \overline{\sigma_{\mathrm{p}}(D_\alpha)} = \sigma(D_\alpha)$.

Problem 7

Let X be a complex Banach space. Let $T \in B(X)$ and let

$$r_\sigma(T) := \sup_{\lambda \in \sigma(T)} |\lambda|$$

denote the **spectral radius** of T. Show that

$$r_\sigma(T) \leqslant \liminf_{n\to\infty} \|T^n\|^{1/n}.$$

Solution

To prove this inequality, it's natural to think about showing $|\lambda| \leqslant \|T^n\|^{1/n}$ or equivalently $|\lambda^n| \leqslant \|T^n\|$. Now suppose $\lambda \in \sigma(T)$, i.e., $T - \lambda I$ is not invertible. Therefore,

$$T^n - (\lambda I)^n = T^n - \lambda^n I = (T - \lambda I) \sum_{i=1}^{n} \left[T^i (\lambda I)^{n-i}\right]$$

is also not invertible by lemma 9.5 since the terms on the RHS commute. Therefore $\lambda^n \in \sigma(T^n)$, and by proposition 8.5 we claim $|\lambda^n| \leqslant \|T^n\|$. Taking the supremum of λ's and letting $n \to \infty$, we conclude that $r_\sigma(T) = \sup_{\lambda\in\sigma(T)} |\lambda| \leqslant \liminf_{n\to\infty} \|T^n\|^{1/n}$.