MATH 580 Problem Set 5

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Problem 1

Suppose that $\{K_{ij}\}_{i,j=1}^{\infty} \subset \mathbb{K}$ is such that $\sum_{i,j=1}^{\infty} |K_{ij}|^2 < \infty$. Show that $S : \ell^2 \to \ell^2$ defined by

$$(S(x))_i := \sum_{j=1}^{\infty} K_{ij} x_j$$

is a compact operator.

Solution

First we verify that S is indeed bounded:

$$\begin{aligned}
\|S(x)\|^2 &= \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} K_{ij} x_j \right|^2 \\
&\leqslant \sum_{i=1}^{\infty} \left[\sum_{j=1}^{\infty} |K_{ij}|^2 \sum_{j=1}^{\infty} |x_j|^2 \right] \\
&= \sum_{i,j=1}^{\infty} |K_{ij}|^2 \|x\|^2 \\
\implies \|S\| &\leqslant \left(\sum_{i,j=1}^{\infty} |K_{ij}|^2 \right)^{1/2} < \infty.
\end{aligned}$$

Furthermore, notice that S is a Hilbert-Schimidt operator: using $\{e^{(i)}\}$ as the orthonormal basis, we have

$$\sum_{i=1}^{\infty} \left\| S(e^{(i)}) \right\|^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |K_{ij}|^2 < \infty \text{ by assumption.}$$

Since ℓ^2 is a separable ($\{e^{(i)}\}$ as the countable set with dense closure) Hilbert (with the canonical inner product) space, by Proposition 9.16 we conclude that S is compact.

Problem 2

(a) Show that the operator $T : \ell^2 \to \ell^2$ that maps $e^{(j)}$ to $e^{(j+1)}$ if j is odd and $e^{(j)}$ to 0 if j is even, i.e., $T(x_1, x_2, \dots) := (0, x_1, 0, x_3, \dots)$, is not compact.

(b) Given an example of $T \in B(X,Y)$ and $S \in B(Y,Z)$, where X, Y, Z are Banach spaces, such that $S \circ T$ is compact, but neither S nor T is. (*This shows that the sufficient condition from Lemma 9.17.1 is not a necessary condition.*)

Solution

(a) For $k \geqslant 1$, taking the bounded sequence $\{e^{(2k-1)}\}$ and applying T, we get $\{e^{(2k)}\}$, among which the pairwise ℓ^2-distance is $\sqrt{2}$. Therefore no subsequence can possibly convergence, and we conclude that T is not compact.

(b) Simply notice that (from above) $T^2 = 0$, trivially compact.

Problem 3

Let X be an infinite dimensional normed space and $T \in K(X)$. Show that there exists a sequence $\{x_n\} \subset S_X$ (the unit sphere) such that $T(x_n) \to 0$. Given an example of such T where there is no $x \in S_X$ such that $T(x) = 0$.

Solution

Pick $x_1 \in X$ with $\|x_1\| = 1$. Since span$\{x_1\}$ is a closed subspace of X, there exists another $x_2 \in X$ such that $\|x_2 - x_1\| \geqslant 1/2$ by Riesz's lemma. Since X is infinite dimensional, we may repeat this process inductively and obtain a sequence $\{x_n\}_{n \geqslant 1}$ such that $\|x_i\| = 1$ for all i's and $\|x_i - x_j\| \geqslant 1/2$ whenever $i \neq j$.
Notice that $\{x_n\} \subset S_X$. If $T \in K(X)$ then there exists a subsequence $\{x_{n_i}\}$, which we now relabel as the new $\{x_n\}$, such that $T(x_n) \to x$ for some $x \in X$. Define the difference sequence $\{y_n\} := \{x_{n+1} - x_n\}$, we have $T(y_n) \to 0$. The convergence of $T(x_n)$ implies $T(y_n) \to 0$.
Let $\epsilon > 0$ be given. There exists $N \in \mathbb{N}$ such that $\|T(y_n)\| < \epsilon/2$ whenever $n \geqslant N$. Recall that by construction each $y_n \geqslant 1/2$. Now we normalize $\{y_n\}$ by defining $\{y_n'\} := \{y_n / \|y_n\|\} \subset S_X$. If $n \geqslant N$, then

$$\|T(y_n')\| = \underbrace{\frac{1}{\|y_n\|}}_{\leqslant 2} \underbrace{T(y_n)}_{<\epsilon/2} < \epsilon,$$

from which we have obtained a sequence on the sphere that converges to 0 under T.

To think of convergence to 0, the most natural idea is to think about $\{1/n\}$. Therefore we consider

$$T : \ell^2 \to \ell^2 \text{ defined by } \{x_n\} \mapsto \{x_n/n\}.$$

(As mentioned in Example 9.20, this operator is clearly bounded with $\|T\| \leqslant 1$.) Now take the sequence $\{e^{(i)}\} \subset S_X$, and we have obtained $\{T(e^{(i)})\} = \{1/n\}$ that converges to 0. Yet, $T(e^{(i)}) = 1/i \neq 0$.

Problem 4

(a) Let X, Y, Z be Banach spaces. Let $K \in K(X,Y)$ and $T \in B(X,Y)$ be injective. Show that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\|K(x)\|_Y \leqslant \epsilon\|x\|_X + C_\epsilon\|T(K(x))\|_Z.$$

This is the so-called Ehrling's Lemma.

Solution

To simplify the inequality, WLOG let $\{x_n\} \subset X$ be a sequence with $\|x_n\| = 1$. Suppose the opposite of the claim is true, then there exists $\epsilon_0 > 0$ such that, for all $n \in \mathbb{N}$,

$$\|K(x_n)\|_Y > \epsilon_0 + n\|TK(x_n)\|_Z.$$

Since $K \in K(X,Y)$, $\{K(x_n)\}$ is bounded. Therefore $\|TK(x_n)\|_Z < (\|K(x_n)\|_Y - \epsilon_0)/n \to 0$. On the other hand, since K is compact, there exists a subsequence $\{x_{n_i}\}$ that converges under K and hence under $T \circ K$ as well. However, since $T \circ K$ is injective (as both T and K are), we have $\|TK(x_n)\|_Z \to 0 \implies x_n \to 0$, contradicting our assumption that $\|x_n\| = 1$ uniformly. ☐

(b) Let $\Omega \in \mathbb{R}^n$ be an open and bounded set with smooth boundary. Show that for all $u \in C^2\left(\overline{\Omega}\right)$

$$\|\nabla u\|_{C(\overline{\Omega})} \leqslant \epsilon\|D^2 u\|_{C(\overline{\Omega})} + C_\epsilon\|u\|_{C(\overline{\Omega})}.$$

Hint: recall a fact from real analysis that if $f_n \to f$ and $\nabla f_n \to g$ uniformly on a compact subset of $\mathbb{R}^n$, then $g = \nabla f$.

(c) Let $u \in C^2([0,1])$ be a solution to the differential equation

$$a\ddot{u} + b\dot{u} + du = 0 \text{ in } (0,1),$$

where $a, b, d \in C([0,1])$ with $a \geqslant c$ for some $c > 0$. Show that form some $C > 0$ (depending only on a, b, c, d)

$$\|u\|_{C^2([0,1])} \leqslant C\|u\|_{C([0,1])}.$$

Problem 5

Let X be a separable normed space. Show that X is isometrically isomorphic to a subspace of ℓ^∞. *Hint: let $\{x_n\}$ be a dense sequence in the unit sphere of X, and consider $T(x) := (\varphi_1(x), \varphi_2(x), \dots)$ where φ is the support functional of x_n.* When is the resulting subspace of ℓ^∞ closed?

Solution

(1) T is linear: trivial, since the support functionals are linear.

(2) T is injective: suppose $T(x) = 0$ for some nonzero x. Since $\{x_n\}$ is dense in S_X, for any $\epsilon > 0$ there exists $x_k \in \{x_n\}$ such that $\|x_k - \frac{x}{\|x\|}\| < \epsilon$.

By triangle inequality $|\varphi_k(x_k)| \leqslant |\varphi_k(x_k - x/\|x\|)| + |\varphi_k(x/\|x\|)|$. Therefore,

$$\left|\varphi_k\left(\frac{x}{\|x\|}\right)\right| \geqslant \underbrace{|\varphi_k(x_k)|}_{=1} - \underbrace{\left|\varphi_k\left(x_k - \frac{x}{\|x\|}\right)\right|}_{\leqslant \|\varphi_k\| \|x_k - x/\|x\|\|}$$
$$\geqslant 1 - 1 \cdot \epsilon.$$

This contradicts with $T(x/\|x\|) = (T(x))/\|x\| = 0$. Hence T is injective.

(3) $\|T\| = 1$: we first show $\|T\| \leqslant 1$:

$$\|T\| = \sup_{\|x\|=1} \|T(x)\| = \sup_{\|x\|=1} \sup_{n\in\mathbb{N}} |\varphi_n(x)| \leqslant \sup_{\|x\|=1} \sup_{n\in\mathbb{N}} \|x\| = 1.$$

On the other hand, from the previous part, letting $\epsilon \downarrow 0$ we have

$$\left\|T\left(\frac{x}{\|x\|}\right)\right\| = \sup_{n\in\mathbb{N}} \left|\varphi_n\left(\frac{x}{\|x\|}\right)\right| \geqslant \left|\varphi_k\left(\frac{x}{\|x\|}\right)\right| \geqslant 1 - \epsilon \to 1.$$

Therefore $\|T\| \geqslant 1$, and the two parts suggest that $\|T\| = 1$.

(4) If X is Banach then the resulting subspace of ℓ^∞ is closed: let $(T(x^{(1)}), T(x^{(2)}), \dots)$ be a sequence in the range(T) that converges to some $y \in \ell^\infty$. Clearly $\{T(x^{(n)})\}$ is Cauchy by triangle inequality. Since T is an isometry, $\|T(x^{(n)}) - T(x^{(m)})\|_{\ell^\infty} \to 0$ as $\min(m,n) \to \infty$ suggests $\|x^{(n)} - x^{(m)}\|_X \to 0$ as well. Assuming X is Banach, $\{x^{(n)}\} \to x$ for some $x \in X$. Then since T is bounded, $T(x^{(n)}) \to T(x)$. By the uniqueness of limit we conclude that $y = T(x) \in \text{range}(T)$, i.e., the range$(T)$ is closed.