MATH 580 Problem Set 6

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Problem 1

Given an example of a normed space X and $f \in X^*$ such that $\|f\|_{X^*} = 1$ but $|f(x)| < \|x\|$ for every $x \in X \setminus \{0\}$. Why does such example not contradict Corollary 10.6 (support functionals)? Can you find such example, but with X reflexive?

Solution

Define X to be $\ell^1(\mathbb{R})$, i.e., the space of all summable sequences, but equipped with ℓ^2 norm. Consider $f \in X^*$ defined by

$$f(x) := \frac{\sqrt{6}}{\pi} \sum_{n=1}^{\infty} \frac{x_n}{n}.$$

Clearly $f \in L(X, \mathbb{R})$, and furthermore f is bounded because, by Cauchy-Schwarz and Basel's $\pi^2/6$ identity,

$$|f(x)| = \left| \frac{\sqrt{6}}{\pi} \sum_{n=1}^{\infty} \frac{x_n}{n} \right| \leqslant \frac{\sqrt{6}}{\pi} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \cdot \|x\|_{\ell^2} = \|x\|_{\ell^2} \implies \|f\| \leqslant 1.$$

On the other hand, we consider the finite truncations of the sequence $\{1/n\}_{n \geqslant 1}$. Let $x^{(n)}$ denote the sequence $(1, 1/2, \ldots, 1/n, 0, \ldots)$. Notice that

$$\|f\| \geqslant \frac{|f(x)|}{\|x\|_{\ell^2}} = \frac{\sqrt{6}}{\pi} \frac{\left| \sum_{i=1}^{n} (1/n)/n \right|}{\sqrt{\sum_{i=1}^{n} |1/n|^2}} = \frac{\sqrt{6}}{\pi} \left(\sum_{i=1}^{n} \frac{1}{n^2} \right)^{1/2}.$$

Taking supremum on the RHS by letting $n \to \infty$ (where the RHS is clearly monotone) gives $\|f\| \geqslant \frac{\sqrt{6}}{\pi} \cdot \frac{\pi}{\sqrt{6}} = 1$. Indeed we have $\|f\| = 1$. However, if it so happens that $|f(x)| = \|x\|$, we see that the $\leqslant$ must be $=$, which can only happen if x is a nonzero scalar multiple of $\{1/n\}_{n \geqslant 1}$. But such $x \notin \ell^1(\mathbb{R})$!

This example does not contradict Corollary 10.6 because that corollary states the *existence* of *some* $f \in X^*$ satisfying those conditions, not that *every* $f \in X^*$ needs to have norm 1 with $|f(x)| = \|x\|$ for some x.

With X reflexive we are no longer to find such example. Suppose there still exists $f \in X^*$ with $\|f\|_{X^*} = 1$

and $|f(x)| < \|x\|$ for every $x \in X \setminus \{0\}$. Then, by the canonical map,

$$|x^{**}(f)| = |f(x)| < \|x\|_X = \|x^{**}\|_{X^{**}} \text{ for all } x^{**} \in X^{**}. \tag{1}$$

On the other hand, treating X^{**} as the dual of X^*, there exists a supporting functional $\Phi \in X^{**}$ such that

$$\|\Phi\|_{X^{**}} = 1 \text{ and } |\Phi(f)| = \|f\|_{X^*} = 1. \tag{2}$$

Since X is reflexive there exists $\varphi \in X$ such that $\varphi^{**} = \Phi$, so $\|\varphi^{**}\|_{X^{**}} = 1$. Since (2) states $|\varphi^{**}(f)| = 1$ but (1) states $|\varphi^{**}(f)| < \|\varphi^{**}\|_{X^{**}} = \|\Phi\|_{X^{**}} = 1$, we have obtained a contradiction. Thus there does not exist $f \in X^*$ satisfying the conditions if X is reflexive.

Problem 2

Let $X := \ell^2(\mathbb{R})$,

$$A := \left\{ \sum_{i=1}^{n} \alpha_i e^{(i)} : n \in \mathbb{N}, \alpha_1, \dots, \alpha_{n-1} \in \mathbb{R}, \alpha_n > 0 \right\} \subset X,$$

and $B := -A$. Show that A, B are disjoint, convex sets, and that $f(A) = f(B) = \mathbb{R}$ for every $f \in X^*$. Why does it not contradict the functional separation theorem?

Solution

(1) Disjoint: if $x := (x_1, x_2, \dots, x_n, 0, \dots) \in A \cap B$, then for any $i \leqslant n$, $x_i > 0$ and $x_i < 0$ simultaneously, clearly a contradiction.

(2) Convex: this follows from the fact that any convex combination of positive numbers is positive and that of negative numbers is negative.

(3) $f(A) = f(B) = \mathbb{R}$: simply notice that $(\ell^2)^* = \ell^2$ (since $1/2 + 1/2 = 1$). Let $f \in (\ell^2)^* = \ell^2$ and $r \in \mathbb{R}$ be given. For any $f = (y_1, y_2, \dots) \in \ell^2 \setminus \{0\}$ (**did you forget to say "nonzero" f, which I believe is a necessary condition?** Otherwise of course $f(A) = f(B) = \{0\}$, not $\mathbb{R}$), there exists at least one nonzero y_n (term in sequence). Now it remains to notice that $f(re^{(n)}/y_n) = ry_n/y_n = r$, so indeed $f(A) = \mathbb{R}$. To show $f(B) = \mathbb{R}$, simple add a negative sign to everything.

(4) This does not violate Theorem 10.16 since A (and B) is (are) not closed. Pick any $a := \sum_{i=1}^{n} \alpha_i e^{(i)} \in A$ and let $\epsilon > 0$ be given. Recall the $\pi^2/6$ identity; we have

$$\left\| a + \sum_{n=1}^{\infty} \frac{6\epsilon e^{(n)}}{n\pi^2} \right\| - \|a\| \leqslant \left\| \sum_{n=1}^{\infty} \frac{6\epsilon e^{(n)}}{n\pi^2} \right\| = \epsilon.$$

Therefore the red term is always in the ball (closed, but can be made open by using 2ϵ) centered at a with radius ϵ. However, this red term is not in A since it is an infinite sequence whereas all elements of A are finite sequences. Hence A is not open, and the Theorem does not apply.

Problem 3

Let X be reflexive and $T \in K(X)$. Suppose that $\{x_n\} \subset X$ is such that

$$c_1 \leqslant \|x_n\| \leqslant c_2 \text{ for all } n \geqslant 1,$$

where $0 < c_1 \leqslant c_2$, and

$$\|T(x_n) - x_n\| \to 0 \text{ as } n \to \infty.$$

Show that T has a nontrivial fixed point, i.e., that $T(x) = x$ for some $x \neq 0$.

Solution

By Theorem 12.10 there exists a subsequence $\{x_{n_k}\}$, which we relabel as $\{x_n\}$, that converges weakly to some $y \in X$. Then Lemma 12.2 says $T(x_n) \to T(y)$ (strongly). On the other hand, by the compactness of T, there exists a sub-subsequence $\{x_{n_k}\}$ (recall we've relabeled the original subsequence as $\{x_n\}$) that converges (strongly) to some $z \in X$ under T. Hence $z = T(y)$. On one hand,

$$\|x_{n_k} - z\| \leqslant \underbrace{\|T(x_{n_k}) - x_{n_k}\|}_{\to 0 \text{ by assumption}} + \overbrace{\|T(x_{n_k}) - z\|}^{\to 0 \text{ by compactness of } T} \to 0, \tag{Δ}$$

which implies $\|z\| \neq 0$ since $\inf \|x_n\| \geqslant c_1 > 0$. On the other hand,

$$\begin{aligned} \|T(z) - z\| &\leqslant \|T(x_{n_k}) - T(z)\| + \|T(x_{n_k}) - z\| \\ &\leqslant \underbrace{\|T\|}_{<\infty} \overbrace{\|x_{n_k} - z\|}^{\to 0 \text{ by } (\Delta)} + \underbrace{\|T(x_{n_k}) - z\|}_{\to 0 \text{ by compactness of } T} \to 0, \end{aligned}$$

so indeed we have found a nontrivial fixed point z.

Problem 4

Let X be reflexive, $A \subset X$ be a closed and convex subset and $x \in X \setminus A$. Show that there exists $a \in A$ such that

$$\|x - a\| = \inf_{a' \in C} \|x - a'\|.$$

Deduce that $X := C([-1, 1])$ is not reflexive.

Remark

This generalizes the existence (and uniqueness) of minimizer in Hilbert spaces (Lemma 6.9) and (more generally) in uniformly convex Banach spaces (PS4.1). Recall Milmlan-Pettis Theorem, i.e., every uniformly convex Banach spaces are reflexive.

Solution

Define $d := \inf_{a' \in C} \|x - a'\|$. It follows that there exists $\{a_n\} \subset A$ such that $\|x - a_n\| \to d$. Again, since $\{a_n\}$ is bounded, by Theorem 12.10 there exists some subsequence, which we now relabel and call it the new $\{x_n\}$, such that $\{x_n\} \rightharpoonup y$ for some $y \in X$. Since A is convex and closed in X, so is $x - A := \{x - a : a \in A\}$, and by Lemma 12.7, $x - A$ is weakly closed. Therefore $y \in x - A$, i.e., there exists some $a \in A$ such that $y_0 = x - a$. Now it remains to show that this $a \in A$ is actually the one we are looking for. Applying the definition if infimum and Lemma 11.3.4 gives

$$d = \inf_{a' \in C} \|x - a'\| = \|x - a\| = \|y\| \leqslant \liminf_{n \to \infty} \|x - a_n\| = d$$

which implies $\|y\| = d$ and $a \in A$ is indeed the minimizer.

$X := C([-1,1])$ is not reflexive; one counterexample is PS 3.7, where $UL = \left\{ g \in X : \int_{-1}^{1} g = \int_{0}^{1} g = 0 \right\}$ is a closed and convex (obvious) subspace of X, but there does not exist a minimizer by PS 3.7(a).

Problem 5

Show that the characterization of weak convergence in ℓ^p spaces from Ex. 12.4.1 (i.e., that $x^{(n)} \rightharpoonup 0$ in ℓ^p if and only if $\{x^{(n)}\}$ is bounded and $x_k^{(n)} \to 0$ for all k) does not hold in ℓ^1 or ℓ^∞, but it does hold for c_0. *Hint: for ℓ^∞ consider Banach limits.*

Solution

(1) ℓ^1: consider $\{e^{(i)}\}$ — clearly bounded and each component converges (something like $(0, \dots, 0, 1, 0, \dots)$) whereas the sequence itself does *not* converge to 0: $\|e^{(i)}\|_{\ell^1} = 1$ for all $i \geqslant 1$. By Schur's theorem this means $e^n \not\rightharpoonup 0$ in ℓ^1.

(2) ℓ^∞: now consider $\{x^{(n)}\} := \left\{ \sum_{i=n}^{\infty} e^{(i)} \right\}_{n \geqslant 1}$ (i.e. the set of sequences of form $(0, \dots, 0, 1, 1, \dots)$, starting from the n^{th} component). Again, this sequence is clearly bounded in $\| \cdot \|_{\ell^\infty}$ and each component converges (of form $(1, \dots, 1, 0, \dots)$). By the hint, since the Banach limit $\lim_{n \to \infty} \mathfrak{L}(x^{(n)}) \geqslant \lim_{n \to \infty} e_n^{(n)} = 1$, the sequence does not converge weakly to 0, where $\mathfrak{L}(0) = 0$.

(3) c_0: since $(c_0)^* = \ell^1$, if we let $E := \operatorname{span}\{e^{(i)}\}$ we see that $E \subset (c_0)^* = \ell^1$ is dense, and the claim follows.

Problem 6

Let X be a real Banach space. A theorem due to James (*Israel J. Math., 1964*) says that if X is not reflexive then there exists $\theta \in (0,1)$ and sequences $\{f_n\} \subset X^*$, $\{x_n\} \subset X$ such that $\|f_n\|_{X^*} = \|x_n\|_X = 1$ for all n and

$$f_n(x_i) \begin{cases} \geqslant \theta & n \leqslant i, \\ = 0 & n > i. \end{cases}$$

Let $C_n := \overline{\text{conv}\{x_n, x_{n+1}, \dots\}}$, i.e., the closure of the set of all convex combinations of $\{x_n, x_{n+1}, \dots\}$.

(a) Suppose that X is not reflexive. Show that $\{C_n\}$ is a nonincreasing sequence of nonempty, closed, bounded, convex sets such that $\bigcap_k C_k = \varnothing$. *Hint: show that if $x \in C_k$ for some k then $f_n(x) \to 0$ as $n \to \infty$, but if $x \in \bigcap_k C_k$ then $f_n(x) \geqslant \theta$ for all n.*

Proof.

(1) Nonincreasing, nonempty, closed: trivial.

(2) Bounded: for any $x := \sum_{i=1}^{k} \lambda_i x_{n_i} \in \text{conv}\{x_n, x_{n+1}, \dots\}$ where $n_1, \dots, n_k \geqslant n$, we have

$$\|x\| \leqslant \sum_{i=1}^{k} \|\lambda_i x_{n_i}\| = \sum_{i=n}^{\infty} \lambda_i = 1$$

so $\text{conv}\{x_n, x_{n+1}, \dots\}$ is bounded. Hence so is C_n the closure.

(3) Convex: for any $x, y \in C_n$, there exist sequences $\{x^{(i)}\}$ and $\{y^{(i)}\}$ in $\text{conv}\{x_n, x_{n+1}, \dots\}$ that converge to x and y, respectively. Let $\lambda \in [0,1]$ be given. It follows that $\lambda x^{(i)} + (1-\lambda) y^{(i)} \in C_n$ by convexness. Therefore taking $i \to \infty$ we see $\lambda x^{(i)} + (1-\lambda) y^{(i)} \to \lambda x + (1-\lambda) y$. Since C_n is closed we conclude that $\lambda x + (1-\lambda) y \in C_n$, and so C_n is convex.

(4) $\bigcap_k C_k = \varnothing$:

(I) Hint 1: let $\epsilon > 0$ be given. Since $x \in \overline{\text{conv}\{x_k, x_{k+1}, \dots\}}$, there exists $\tilde{x}$ in the set of convex combinations such that $\|x - \tilde{x}\| < \epsilon$. Suppose $\tilde{x} = \sum_{i=1}^{j} \lambda_i x_{k_i}$ where the λ's add up to 1 and x_{k_i} are from $\{x_k, x_{k+1}, \dots\}$. It follows that whenever $n > \max(x_{k_1}, \dots, x_{k_j})$ we have

$$f_n(\tilde{x}) = \sum_{i=1}^{j} \lambda_i \underbrace{f_n x_{k_i}}_{\substack{=0 \\ \text{by assumption}}} = 0$$

whereas, by triangle inequality and assumptions on $\|f_n\|$,

$$|f_n(x)| \leqslant |f_n(x - \tilde{x})| + |f_n(\tilde{x})| \leqslant \|f_n\| \|x - \tilde{x}\| + \|f_n(\tilde{x})\| < \epsilon.$$

Therefore $f_n(x) \to 0$, as desired.

(I) Hint 2: for any n, consider $k > n$. Let $x \in C_k$ and pick $\tilde{x} := \sum_{i=1}^{j} \lambda_i x_{k_i}$ from the set of convex combinations of $\{x_k, x_{k+1}, \ldots\}$ with $x \neq \tilde{x}$. Then

$$f_n(\tilde{x}) = \sum_{i=1}^{j} \lambda_i \underbrace{f_n(x_{k_i})}_{\substack{\geqslant \theta \\ \text{by construction}}} \geqslant \theta.$$

Therefore $f_n(x) = f_n(x - \tilde{x}) + f_n(\tilde{x}) \geqslant f_n(\tilde{x}) \geqslant \theta$.

Having shown both hints, we see that no x meets both hints, Hence $\bigcap_k C_k = \varnothing$.

□

(b) Now suppose that every bounded sequence in X has a weakly convergent subsequence. Show that if $\{C_n\}$ is a nonincreasing sequence of nonempty, closed, bounded, convex subsets of X then $\bigcap_k C_k \neq \varnothing$. *Hint: use the characterization of convex setse in terms of the envelope of supporting hyperplanes.*

Proof. Pick $x_n \in C_n$ for each n. Since the C_n's are bounded by (a), here by assumption we have a weakly convergent subsequence $\{x_{n_k}\}$ that converges weakly to some $x \in X$. By the hint, since $x_{n_1} \in C_{n_1}$, for all $f \in X^*$

$$f(x_{n_1}) \geqslant \inf_{y \in C_{n_1}} f(y).$$

By weak convergence $f(x_{n_1}) \to f(x_0)$. Since $C_{n_1} \supset C_{n_2} \supset \ldots$ we see that $x_{n_1}, x_{n_2}, \ldots$ are also in C_{n_1}. Therefore $f(x_{n_i}) \geqslant \inf f(y)$ for all $i \geqslant 1$, and we see $f(x) \geqslant \inf f(y) \implies x \in C_{n_1}$. Likewise $x \in C_{n_i}$ for all $i \geqslant 1$. Once again, since the C_n's (not just C_{n_i}'s but all of them) are nested, we see that $x \in \bigcap_k C_k$; hence $\bigcap_k C_k \neq \varnothing$. □

(c) Deduce from (a) and (b) that if the closed unit ball $\overline{B_X(0,1)} \subset X$ is weakly compact then X is reflexive. *This gives the $\Longleftarrow$ implication in Theorem 12.10 for real Banach spaces.*

Proof. Since the unit ball is weakly compact, every bounded sequence in X has a weakly convergent subsequence. By (b), if $\{C_n\}_{n \geqslant 1}$ is a *noncreasing sequence of nonempty, closed, bounded, convex subsets of* X then $\bigcap_k C_k \neq \varnothing$, whereas if X is nonempty, the contrary is also true by (a). Thus X must be reflexive! □