MATH 408 Homework 4

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Problem 1

Let $X_1, \ldots, X_n$ be a random sample of size n from a Poisson distribution with unknown parameter $\lambda > 0$. Let Y be the estimator $Y = \chi_{\{X_1=0\}}$. Suppose we want to estimate $e^{-\lambda}$.

(1) Find a method of moments estimator for $e^{-\lambda}$. Is this estimator consistent?

(2) Show that Y is unbiased for $e^{-\lambda}$.

(3) Show that $\sum_{i=1}^{n} X_i$ is sufficient for $e^{-\lambda}$.

(4) Compute $W_n := \mathbb{E}_\lambda(Y \mid \sum_{i=1}^n X_i)$ as in Rao-Blackwell.

(5) As $n \to \infty$, does W_n congerge in any sense? If so, what does it converge to? Does this mean that $W_1, W_2, \ldots$ is consistent?

Solution. (1) Since $\mathbb{E}X = \sum_{k=0}^{\infty} k \frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=1}^{\infty} k \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda}\lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = e^{-\lambda}\lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda}\lambda e^{\lambda} = \lambda$, one method of moments estimator for $e^{-\lambda}$ is given by

$$\exp\left(-\frac{1}{n}\sum_{i=1}^{n} X_i\right).$$

Since $\frac{1}{n}\sum_{i=1}^{n} X_i$ is consistent, so is our estimator, a continuous image.

(2) This is because $\mathbb{E}Y = 0 \cdot \mathbb{P}(X_1 \neq 0) + 1 \cdot \mathbb{P}(X_1 = 0) = e^{-\lambda}\lambda^0/0! = e^{-\lambda}$.

(3) By factorization theorem,

$$\mathbb{P}_\lambda(x_1, \ldots, x_n) = \prod_{i=1}^{n} \mathbb{P}_\lambda(x_i) = \frac{e^{-n\lambda}\lambda^{x_1+\ldots+x_n}}{(x_1!)\ldots(x_n!)} = \underbrace{e^{-\lambda n}\lambda^{\Sigma X_i}}_{h_\lambda(\Sigma X_i)} \cdot \underbrace{(\prod_{i=1}^{n} x_i!)^{-1}}_{h(x)}$$

so $\sum_{i=1}^{n} X_i$ is sufficient for λ (and therefore for $e^{-\lambda}$).

(4) By definition $W_n = g(K)$ where $g(k) := \mathbb{E}_\lambda(Y \mid \sum_{i=1}^n X_i = k)$. Also note that summing (finitely many) independent Poisson random varaibles results in another Poisson random variable with parameter $\sum \lambda$.

Hence

$$\begin{aligned}\mathbb{E}_\lambda(Y \mid \sum_{i=1}^n X_i = k) &= 1 \cdot \mathbb{P}(Y = 0 \mid \sum_{i=1}^n X_i = k) + 0 \cdot \mathbb{P}(Y = 1 \mid \sum_{i=1}^n X_i = k) \\ &= \frac{\mathbb{P}(X_1 = 0, \sum_{i=1}^n X_i = k)}{\mathbb{P}(\sum_{i=1}^n X_i = k)} = \frac{\mathbb{P}(X_1 = 0)\mathbb{P}(\sum_{i=2}^n X_i = k)}{\mathbb{P}(\sum_{i=1}^n X_i = k)} \\ &= \frac{\mathbb{P}(\text{Poisson}(\lambda) = 0)\mathbb{P}(\text{Poisson}((n-1)\lambda) = k)}{\mathbb{P}(\text{Poisson}(n\lambda) = k)} \\ &= \frac{(e^{-\lambda})(e^{-n\lambda+\lambda}(n-1)^k\lambda^k/k!)}{e^{-n\lambda}n^k\lambda^k/k!} \\ &= \frac{e^{-n\lambda}(n-1)^k\lambda^k k!}{e^{-n\lambda}n^k\lambda^k k!} = \left(\frac{n-1}{n}\right)^k\end{aligned}$$

and so $\mathbb{E}_\lambda(Y \mid \sum_{i=1}^n X_i = k) = \left(\frac{n-1}{n}\right)^{\sum_{i=1}^n X_i}$.

(5) As $n \to \infty$, for each k, $\mathbb{E}_\lambda(Y \mid \sum_{i=1}^n X_i = k)$ converges to 1. Hence $W_n \to$ the constant random variable 1almost surely. The W_n's are not consistent as $e^{-\lambda} \neq 1$ for $\lambda > 0$.

Problem 2

Let $X_1, \ldots, X_n$ be a random sample of size n from the uniform distribution on $[0, \theta]$ where $\theta > 0$ is unknown. On a previous homework, we showed that $X_{(n)}$ is a sufficient statistic for θ.

(1) Show that $2X_1$ is unbiased for θ.

(2) Compute $W := \mathbb{E}_\theta(2X_1 \mid X_{(n)})$ as in Rao-Blackwell. Show that W is unbiased for θ.

(3) A method of moments estimator for θ is $\frac{2}{n}\sum_{i=1}^n X_i$. Compute

$$\mathbb{E}_\theta\left(\frac{2}{n}\sum_{i=1}^n X_i \mid X_{(n)}\right).$$

Solution. (1) Clearly, for a uniform distribution, $\mathbb{E}X_1 = \theta/2$ so $2X_1$ is unbiased for θ.

(2) We consider $\mathbb{E}_\theta(2X_1 \mid X_{(n)} = k)$. There is a $1/n$ probability that $X_1 = X_{(n)}$; for the remaining probability, $X_1 < X_{(n)}$, in which case X_1 is uniform on $(0, k)$, and the corresponding expected value is $k/2$. Hence

$$W(k) := \mathbb{E}_\theta(2X_1 \mid X_{(n)} = k) = 2k \cdot \mathbb{P}(X_1 = X_{(n)}) + 2 \cdot \frac{k}{2} \cdot \mathbb{P}(X_1 < X_{(n)}) = \frac{2k}{n} + \frac{k(n-1)}{n} = \frac{k(n+1)}{n}.$$

Thus by definition $W = X_{(n)} \cdot \frac{n+1}{n}$. Using total expectation theorem, $\mathbb{E}W = \mathbb{E}[\mathbb{E}_\theta(2X_1 \mid X_{(n)})] = \mathbb{E}_\theta(2X_1) = \theta$, so W is indeed unbiased.

(3) If $X_{(n)} = k$ then by linearity of conditional expectation

$$\mathbb{E}_\theta\left(\frac{2}{n}\sum_{i=1}^n X_i \mid X_{(n)} = k\right) = \frac{1}{n} \cdot \sum_{i=1}^n \mathbb{E}_\theta(2X_i \mid X_{(n)} = k) = \frac{k(n+1)}{n}$$

so the answer is identical as above.

Problem 3

Let $X_1, \dots, X_n$ be a random sample of size n from a Bernoulli distribution with parameter $\theta \in (0,1)$ unknown. In class we showed that $\sum_{i=1}^{n} X_i$ is sufficient for θ and that

$$\mathbb{E}_\theta(X_1 \mid \sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

That is, Rao-Blackwell suggests that the sample mean has small variance among unbiased estimators for θ.

(1) Compute the Fisher information $I_{X_1}(\theta)$.

(2) Compute the Fisher information $I_{(X_1,\dots,X_n)}(\theta)$.

(3) Show that $\operatorname{var}(n^{-1}\sum_{i=1}^{n} X_i) = \theta(1-\theta)/n$.

(4) Does the sample mean $n^{-1}\sum_{i=1}^{n} X_i$ achieve equality in Cramér-Rao? If so, then $n^{-1}\sum_{i=1}^{n} X_i$ is UMVU.

Solution. (1) For $x \in \{0,1\}$, the PMF of X is given by $\mathbb{P}(X = x) = \theta^x(1-\theta)^{1-x}$. Since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}\log(\theta^x(1-\theta)^{1-x}) &= \frac{1}{\theta^x(1-\theta)^{1-x}}(x\theta^{x-1}(1-\theta)^{1-x} - \theta^x(1-x)(1-\theta)^{-x}) \\ &= \frac{x\theta^{x-1}(1-\theta)^{1-x}}{\theta^x(1-\theta)^{1-x}} - \frac{\theta^x(1-x)(1-\theta)^{-x}}{\theta^x(1-\theta)^{1-x}} \\ &= \frac{x}{\theta} - \frac{1-x}{1-\theta},\end{aligned}$$

we see that

$$\begin{aligned}I_{X_1}(\theta) &= \mathbb{E}\left[\frac{X}{\theta} - \frac{1-X}{1-\theta}\right]^2 \\ &= \mathbb{E}\left[\frac{X^2}{\theta^2}\right] - 2\mathbb{E}\left[\frac{X(1-X)}{\theta(1-\theta)}\right] + \mathbb{E}\left[\frac{(1-X)^2}{(1-\theta)^2}\right] \\ &= \mathbb{E}\left[\frac{X^2}{\theta^2}\right] - 2\mathbb{E}\left[\frac{X-X^2}{\theta(1-\theta)}\right] + \mathbb{E}\left[\frac{X^2-2X+1}{(1-\theta)^2}\right] \\ &\overset{\Delta}{=} \frac{\theta}{\theta^2} - 2\cdot\frac{\theta-\theta}{\theta(1-\theta)} + \frac{\theta-2\theta+1}{(1-\theta)^2} = \frac{1}{\theta} + \frac{1-\theta}{(1-\theta)^2} = \frac{1}{\theta(1-\theta)}.\end{aligned}$$

(The Δ step is because $\mathbb{E}X = p$ and $\mathbb{E}X^2 = p$, both of which follow from simple calculation.)

(2) Since $X_1, \dots, X_n$ are i.i.d. (in particular independent),

$$I_{(X_1,\dots,X_n)}(\theta) = \sum_{i=1}^{n} I_{X_i}(\theta) = \frac{n}{\theta(1-\theta)}.$$

(3) This also follows from definition: since $X_1, \dots, X_n$ are i.i.d. (in particular independent),

$$\operatorname{var}(n^{-1}\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\operatorname{var}(X_i) = \frac{\theta(1-\theta)}{n}.$$

(4) Yes.