Beginning of Aug.23, 2021

0.1 Review of Probability

Definition 0.1.1: Axioms of Probability

Assume there exists some (nonempty) *universal set* Ω that contains all other sets (events). We denote $\mathbb{P}$ as a probability law on Ω. The following are the three axioms of $\mathbb{P}$:

(1) For all subsets $A \subset \Omega$, $0 \leqslant \mathbb{P}(A) \leqslant 1$.

(2) $\mathbb{P}$ is (countably) additive for *disjoint* sets, i.e., if $\{A_n\}$ are disjoint then $\mathbb{P}(\bigcup_{i=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.

(3) $\mathbb{P}(\Omega) = 1$.

From (2) we immediately see that $\mathbb{P}(A \cup B) \leqslant \mathbb{P}(A) + \mathbb{P}(B)$ and = can be obtained if and only if $A \cap B = \varnothing$.

Definition 0.1.2: Conditional Probability

If $A, B \subset \Omega$, we define the **conditional probability** as

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Definition 0.1.3: Continuous Random Variable, CRV

A **random variable** is a function $X : \Omega \to \mathbb{R}$. We say X is **continuous** if

$$\mathbb{P}(a \leqslant X \leqslant b) = \int_a^b f_X(x)\,\mathrm{d}x$$

for some $f_X : \mathbb{R} \to [0, \infty)$ and for all $-\infty \leqslant a \leqslant b \leqslant \infty$. We call f_X the **PDF, probability density function**. We also define the **CDF, cumulative density function**, by $F_X(t) = \mathbb{P}(X \leqslant t)$.

Example 0.1.4. Suppose X is uniformly distributed in $[0, 1]$. Then for any $0 \leqslant a \leqslant b \leqslant 1$,

$$\mathbb{P}(a \leqslant X \leqslant b) = \int_a^b \mathrm{d}x = b - a,$$

indeed a CRV.

Definition 0.1.5: Independence of Finitely Many Sets

Let $A_1, \ldots, A_n \subset \Omega$. We say they are **independent** if for any $S \subset \{1, 2, \ldots, n\}$,

$$\mathbb{P}(\bigcap_{i \in S}(A_i)) = \prod_{i \in S} \mathbb{P}(A_i).$$

Definition 0.1.6: Independence of Countably Many Sets

We say $\{A_n\}$ are independence if, for all $n \geqslant 1$, $A_1, \ldots, A_n$ are independent.