Beginning of Aug.25, 2021

0.1 Some Random Variables

Example 0.1.1: Bernoulli. Let $0 < p < 1$. Define a random variable by $\mathbb{P}(X = 0) = 1 - p$ and $\mathbb{P}(X = 1) = p$.

Example 0.1.2: Binomial. Let $n \in \mathbb{N}$. For $0 \leqslant k \leqslant n$, let $\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$. *"Number of heads flipped among n biased coin flips."*

Example 0.1.3: Geometric. For $k \in \mathbb{N}$, define $\mathbb{P}(X = k) = (1-p)^{k-1}p$. *"Number of coin clips needed to see heads for the first time."*

Definition 0.1.4: Normal Random Variable

Let $\mu \in \mathbb{R}$ and $\sigma > 0$ be two parameters. A random variable X is said to be **normal** or **Gaussian** if X has the X has pdf

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Definition 0.1.5: Gamma Function

For all $\alpha > 0$, define the **Gamma function**

$$\Gamma(\alpha) := \int_0^\infty e^{-x} x^{\alpha-1} \, \mathrm{d}x.$$

Integration by parts suggests that Γ interpolates the factorial: $\Gamma(1) = 1$ and $\Gamma(n+1) = (n+1)\Gamma(n)$.

Definition 0.1.6: Gamma Distribution & Chi-Squared Distribution

Let $\alpha, \beta > 0$. We say X is an (α, β) distributed **Gamma random variable** if X has the pdf

$$f_X(x) = \frac{x^{\alpha-1}\exp(-x/\beta)}{\beta^\alpha \Gamma(\alpha)} \cdot \chi_{[0,\infty)}(x). \tag{1}$$

For example, if $\alpha = p/2$ and $\beta = 2$, we get a **chi-squared** distribution. Its pdf with p **degrees of freedom** is

$$f_X(x) = \frac{x^{p/2-1}\exp(-x/2)}{2^{p/2}\Gamma(p/2)} \cdot \chi_{[0,\infty)}(x). \tag{2}$$

(2) is the distribution of a sum of p independent, squared, standard Gaussian distributions $(\mu = 0, \sigma = 1)$.

Definition: (1.36) Indicator Functions

Let $A \subset \Omega$. We define the **indicator function** $\chi_A : \Omega \to \{0,1\}$ by

$$\chi_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A. \end{cases}$$

Definition: (1.37) Expected Values

Let $\mathbb{P}$ be a probability law on Ω. Let $X : \Omega \to [0,\infty)$ be a (nonnegative) random variable. The **expected value** is defined by

$$\mathbb{E}(X) := \int_0^\infty \mathbb{P}(X > t)\, \mathrm{d}t.$$

If $X : \Omega \to \mathbb{R}$ and if $\mathbb{E}|X| < \infty$, define

$$\mathbb{E}(X) := \mathbb{E}(\max(X,0)) - \mathbb{E}(\max(-X,0))$$

If X is discrete, $\mathbb{E}(X) = \sum_{k\in\mathbb{R}} k \cdot \mathbb{P}(X = k)$.

If X is continuous with PDF f_X, $\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, \mathrm{d}x$.

Proposition: (1.43)

The expected value of (finite) sums is the (finite) sum of expected values:

$$\mathbb{E}\Big(\sum_{i=1}^n X_i\Big) = \sum_{i=1}^n \mathbb{E}(X_i).$$

Definition 0.1.7: Variance & Standard Deviation

The **variance** of a random variable is defined by

$$\mathrm{Var}(X) := \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \mathbb{E}(X - \mathbb{E}(X))^2.$$

and the **standard deviation** is defined to be the square root of above.

Important property of variance: $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(x)$

Definition 0.1.8: Joint PDF

Let X, Y be random variables and let $f_{X,Y} : \mathbb{R}^2 \to [0,\infty)$ be their **joint pdf**. Then

$$\mathbb{P}(a \leqslant X \leqslant b, c \leqslant Y \leqslant d) = \int_a^b \int_c^d f_{X,Y}(x,y)\, \mathrm{d}y\, \mathrm{d}x$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, \mathrm{d}y\, \mathrm{d}x = 1.$$

Definition 0.1.9: Marginal Densities

Continuing on the previous example, the **marginal** of X is given by

$$f_X(x) := \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y,$$

i.e., we "fix" x and integrate $f_{X,Y}$ over all possible values of y. Likewise for $f_Y(y)$.

The density of the conditional $X \mid Y = y$ is

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

Corollary 0.1.10

$\mathbb{E}(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{x,y}(X,Y)\,\mathrm{d}x\,\mathrm{d}y$. Similarly, if $g : \mathbb{R}^2 \to \mathbb{R}$, then

$$\mathbb{E}(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Definition 0.1.11: Independence of Random Variables

Let $X_1, \ldots, X_n : \Omega \to \mathbb{R}$. We say they are **independent** if

$$\mathbb{P}_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \prod_{i=1}^{n} \mathbb{P}_{X_i}(x_i) \qquad x_i \in \mathbb{R}.$$