MATH 408 Quiz 1 Solution Sketch

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Q1

Solution. Let each person flip n flips first. Define

$$E_A := \{\text{events where person 1 has more heads}\}$$
$$E_B := \{\text{events where person 2 has more heads}\}$$
$$E_0 := \{\text{event where both have same number of heads}\}.$$

Clearly $\mathbb{P}(E_A) + \mathbb{P}(E_B) + \mathbb{P}(E_0) = 1$. Note that if E_A happens then person 1 is guaranteed to have more heads after an additional flip; if E_0 happens, person 1 has probability 0.5 to; and if E_B happens, person 1 cannot have more heads. Also, since the coin is fair, $\mathbb{P}(E_A) = \mathbb{P}(E_B)$. Thus, after an additional flip,

$$\begin{aligned}\mathbb{P}(\text{person 1 has more heads}) &= \mathbb{P}(E_A) + \mathbb{P}(E_0)/2 \\ &= \mathbb{P}(E_A) + \frac{1 - \mathbb{P}(E_A) - \mathbb{P}(E_B)}{2} \\ &= \mathbb{P}(E_A) + \frac{1 - 2\mathbb{P}(E_A)}{2} = \frac{1}{2}.\end{aligned}$$

Q2

Proof. We first compute the joint CDF F_{X_2,Y_2} in terms of F_{X_1,Y_1}.

$$F_{X_2,Y_2}(x,y) = \mathbb{P}(X_2 \leqslant x, Y_2 \leqslant y).$$

Since T, S are bijective maps, there exists $(x', y') := T^{-1}(x,y) = S(x,y) \in \mathbb{R}$ such that

$$X_2 \leqslant x, Y_2 \leqslant y \text{ if and only if } X_1 \leqslant x', Y_1 \leqslant y'.$$

Therefore

$$F_{X_2,Y_2}(x,y) = \mathbb{P}(X_2 \leqslant x, Y_2 \leqslant y) = \mathbb{P}(X_1 \leqslant x', Y_1 \leqslant y') = F_{X_1,Y_1}(x', y')$$

subject to $(x', y') = S(x,y)$. Differentiating gives

$$f_{X_2,Y_2}(x,y) = f_{X_1,Y_1}(x',y')|\mathcal{J}(x,y)| = f_{X_1,Y_1}(S(x,y))|\mathcal{J}(x,y)|,$$

as desired. □

Q3

Solution. Let X_i be the indicator variable that evaluates to 1 if the i^{th} person to arrive is hired. Then, among the first i people to arrive, since each permutation is equally likely to occur, $\mathbb{E}X_i = 1/i$. Clearly $\sum_{i=1}^{n} X_i$ is the random variable whose output represents the number of people hired, so it remains to compute its expectation:

$$\mathbb{E}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathbb{E}X_i = \sum_{i=1}^{n} \frac{1}{i}.$$

Q4

Solution. Let X_i be the random variable that evaluates to 1 if the i^{th} flip is heads and 0 otherwise. Then clearly $X_1, X_2, \ldots$ are i.i.d. with $\mu := \mathbb{E}X_1 = 1/2$, $\operatorname{var}(X_1) = 1/4$, and $\sigma := \sqrt{\operatorname{var}(X_1)} = 1/2$. We want to know the probability of > 501000 heads, so it suffices to compute its complement, that of $\leqslant 501000$ heads. CLT gives

$$\mathbb{P}(\frac{X_1 + \ldots + X_n - n\mu}{\sigma\sqrt{n}} \leqslant t) \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp(-s^2/2)\, \mathrm{d}s.$$

Since the event $\{\leqslant 501000 \text{ heads}\}$ correspond to $\sum_{i=1}^{10^6} X_i \leqslant 501000$, plugging in $n = 10^6, \mu = \sigma = 0.5$ suggests that we should set $t = (501000 - 500000)/(1/2 \cdot \sqrt{10^6}) = 2$. That is,

$$\begin{aligned} \mathbb{P}(\leqslant 501000 \text{ heads}) &= \mathbb{P}(\sum X_i \leqslant 501000) = \mathbb{P}(\frac{\sum X_i - n\mu}{\sigma\sqrt{n}} \leqslant 2) \\ &\approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{2} \exp(-s^2/2)\, \mathrm{d}s \approx 0.9772. \end{aligned}$$

Hence, the probability of getting > 501000 heads is approximately $1 - 0.9772 = 0.0228$.

Q5

Solution. Let N_i and N be defined as suggested by the hint. Then $N_1 = 1$. Also define $N_0 = 0$. Now we compute $N_i - N_{i-1}$. If our collection now contains $i - 1$ distinct cards, the probability of buying and getting a new card is $(100 - (i - 1))/100$. Therefore the expected number of cards to buy in order to get a new card is $100/(100 - i + 1)$. Thus,

$$\begin{aligned} \mathbb{E}N = \mathbb{E}N_{100} &= \sum_{i=1}^{100} \mathbb{E}(N_i - N_{i-1}) \\ &= 100 \sum_{k=0}^{99} \frac{1}{100 - k} = 100 \sum_{k=1}^{100} \frac{1}{k} \approx 518.7. \end{aligned}$$

That is, we need to buy $51.87 \approx 52$ packs in order to get a complete set of cards.

Q6

Solution. Following the hint,

$$\begin{aligned} \mathbb{E}(X \mid Y = 1) &= 3 \\ \mathbb{E}(X \mid Y = 2) &= 7 + \mathbb{E}X \\ \mathbb{E}(X \mid Y = 3) &= 9 + \mathbb{E}X. \end{aligned}$$

Then, since $\mathbb{E}X = \sum_{i=1}^{3} \mathbb{P}(Y = i)\mathbb{E}(X \mid Y = i)$, we obtain the function

$$\frac{3 + 7 + \mathbb{E}X + 9 + \mathbb{E}X}{3} = \mathbb{E}X \implies \mathbb{E}X = 19.$$

Q7

Solution. Following the hint, let $X_i = 1$ if the i^{th} sampled person has a savings account and 0 otherwise. It follows from the question that X_i'are i.i.d. with $\mu := \mathbb{E}X_1 = 1/25$, $\operatorname{var}(X_1) = 24/625$, and $\sigma := \sqrt{\operatorname{var}(X_1)} = \sqrt{24/625}$. Using the CLT, we see that $\sum_{i=1}^{625} X_i$ is approximately a Gaussian with mean $n\mu = 25$ and standard deviation $\sigma\sqrt{n} = \sqrt{24}$. Therefore, assuming $\mathbb{P}(\mathbb{Z} \in [-2, 2]) \approx 0.95$ for a standard Gaussian Z, we obtain the confidence interval $[25 - \sqrt{24}, 25 + \sqrt{24}] \approx [20.10, 29, 90]$.

Q8

Solution. We hypothesize that $p = 18/38$. Let X_i be defined as stated in the problem. It follows that they are i.i.d. with $\mathbb{E}X_1 = 18/38$, $\operatorname{var}(X_1) = (18\cdot 20)/38^2 = 360/38^2$, and $\sigma := \sqrt{\operatorname{var}(X_1)} = \sqrt{360/38^2}$. Since $n = 3800$, the random variable $\sum_{i=1}^{3800} X_i$ is approximately a Gaussian with mean $n\mu = 1800$ and standard deviation $\sigma\sqrt{n} = \sqrt{360} \approx 18.97$. But then $1868 - 1800 > 2 \cdot 18.97$ so we indeed reject the null hypothesis that $p = 18/38$. The wheel is therefore unbiased (at least from the experiment we have conducted).

Q9

Solution. For method 1, there is a probability of n_i/m to choose a family of i children. Then there is a probability of $1/i$ that the children picked is the first-born. Thus, the total probability is $m^{-1} \sum_{i=1}^{k} n_i/i$.

For the second method, we simply need to compute the number of first-born children and decide it by the total number of children. Clearly m family correspond to m first-born children, and there are $\sum_{i=1}^{k} i \cdot n_i$ children. Thus the total probability is $m / \sum_{i=1}^{k} i \cdot n_i$.