Q1

Proof. Notice that we have another definition for $Y(s)$:

$$Y(s) = \sup\{t \in \mathbb{R} : F(t) < s\} = \inf\{t \in \mathbb{R} : F(t) \geqslant s\}. \tag{1}$$

Furthermore, by definition of supremum and infimum, whether or not the inequalities are strict impose no effect, so $\leqslant$ and $<$, $\geqslant$ and $>$ are freely interchangeable.
Now, given $F : \mathbb{R} \to [0,1]$, the CDF of X, let $Y : \mathrm{Range}(F) \to \mathbb{R}$ be its generalized inverse. Then by (1) we have

$$Y(F(t)) = \inf\{\tilde{t} \in \mathbb{R} : F(\tilde{t}) \geqslant F(t)\} \leqslant t$$

since t is in the set of which the infimum is taken. A symmetric argument for $F(Y(t))$ can be obtained analogously, and thus

$$Y(F(t)) \leqslant t \qquad \text{and} \qquad F(Y(s)) \geqslant s. \tag{2}$$

Also observe that Y is monotone increasing: if $a \leqslant b$ then

$$\{x : F(x) \geqslant b\} \subset \{x : F(x) \geqslant a\}$$

so

$$\inf\{x : F(x) \geqslant b\} = Y(b) \geqslant Y(a) = \inf\{x : F(x) \geqslant a\}. \tag{3}$$

Now we prove $\mathbb{P}(Y \leqslant t) = F(t)$. This is true because on one hand

$$\begin{aligned}
\mathbb{P}(Y \leqslant t) &= \mathbb{P}_{\text{unif}}(\{s \in [0,1] : Y(s) \leqslant t\}) \\
&= \mathbb{P}_{\text{unif}}(\{s \in [0,1] : F(Y(s)) \leqslant F(t)\}) \\
&\leqslant \mathbb{P}_{\text{unif}}(\{x \in [0,1] : s \leqslant F(t)\}) && \text{[By (2)]} \\
&= \int_0^{F(t)} 1 \, \mathrm{d}s = F(t),
\end{aligned}$$

and on the other hand

$$\begin{aligned}
F(t) = \mathbb{P}(X \leqslant t) &= \mathbb{P}_{\text{unif}}(\{s \in [0,1] : s < F(t)\}) \\
&= \mathbb{P}_{\text{unif}}(\{s \in [0,1] : Y(s) < Y(F(t))\}) \\
&\leqslant \mathbb{P}_{\text{unif}}(\{s \in [0,1] : Y(s) < t\}) && \text{[By (2)]} \\
&= \mathbb{P}_{\text{unif}}(\{s \in [0,1] : Y(s) \leqslant t\}) && [\mathbb{P}(Y(s) = t) = 0] \\
&= \int_0^t Y(s) \, \mathrm{d}s = \mathbb{P}(Y \leqslant t).
\end{aligned}$$

□

Q2

Proof. Notice that the inverse transformations are given by

$$U_1 = \exp\left(-\frac{X^2 + Y^2}{2}\right) \qquad \text{and} \qquad U_2 = \frac{1}{2\pi} \arctan(Y/X).$$

(The first is obtained by taking $X^2 + Y^2$ to cancel out U_2 and the second is by taking Y/X to cancel out U_1.)

Then, the Jacobian for the transformation $(X, Y) \mapsto (U_1, U_2)$ is

$$\begin{aligned}
\begin{vmatrix} \partial U_1/\partial X & \partial U_1/\partial Y \\ \partial U_2/\partial X & \partial U_2/\partial Y \end{vmatrix} &= \begin{vmatrix} -\exp()X & -\exp()Y \\ -\frac{1}{2\pi}\frac{1}{1+Y^2/X^2}\frac{Y}{X^2} & \frac{1}{2\pi}\frac{1}{1+Y^2/X^2}\frac{1}{X} \end{vmatrix} \\
&= \left| -\exp\left(-\frac{X^2+Y^2}{2}\right)\frac{1}{2\pi}\frac{1}{1+Y^2/X^2}\left(1+\frac{Y^2}{X^2}\right) \right| \\
&= \frac{1}{2\pi}\exp\left(-\frac{X^2+Y^2}{2}\right).
\end{aligned}$$

Therefore,

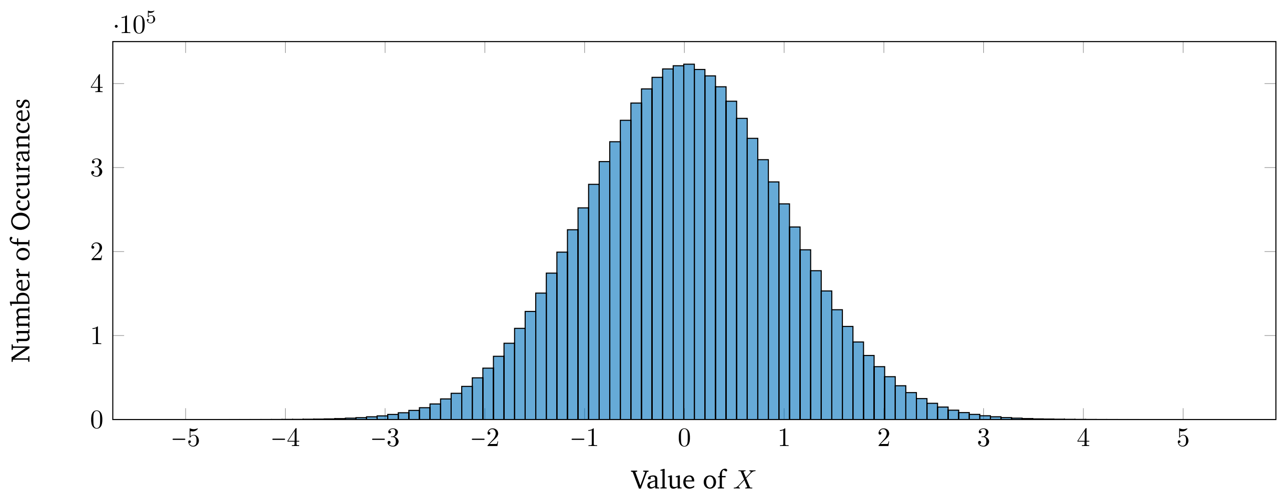
$$\begin{aligned}
f_{X,Y}(x,y) &= f_{U_1,U_2}(u_1,u_2)\mathcal{J}(u_1,u_2) \\
&= 1 \cdot \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}}e^{-y^2/2}.
\end{aligned}$$

A simple calculation shows that the X-marginal and Y-marginal indeed have the PDFs of a Gaussian, and the claim therefore follows as $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. □

Code and output for X below:

```
U1 = rand(1,10^7);
U2 = rand(1,10^7);
X = sqrt((-2 * log(U1))) .* sin(2*pi*U2);
Y = sqrt((-2 * log(U1))) .* cos(2*pi*U2);

histogram(X,100);
```



Q3

Solution. According to the Delta method, if $\sqrt{n}(Y_n - 0)$ converges to $\mathcal{N}(0, 1/4)$ in distribution then

$$\sqrt{n}(f(Y_n) - f(0))$$

converges in distribution to $\mathcal{N}(0, f'(0)^2/4) = \mathcal{N}(0, \cos(2)^2/4)$. Then by identity $\mathbb{E}(Z^4) = 3\sigma^4$ we obtain $\mathbb{E}(Z^4) = 3\cos^4(2)/16$.

Q4

Solution. According to the second-order Delta method, since $\sqrt{n}(Y_n - 0)$ converges in distribution to $\mathcal{N}(0, 1/4)$, $f'(0) = 0$, and $f''(0) = 1 \neq 0$, we see that

$$n(f(Y_n) - f(0))$$

converges in distribution to $\chi_1^2 \cdot \sigma^2 f''(0)/2 = \chi_1^2/4$. Thus $\mathbb{E}[Z] = \mathbb{E}[\chi_1^2]/4 = 1/4$.

Q5

Solution. It is clear that in this case

$$\alpha = (\mathbb{E}X)^2/\mathbb{E}X^2 \qquad \text{and} \qquad \beta = \mathbb{E}X^2/\mathbb{E}X,$$

so the method of moments gives

$$\hat{\alpha} = \left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2 \left(\frac{1}{n}\sum_{i=1}^n X_i^2\right)^{-1} = \frac{1}{n}\left(\sum_{i=1}^n X_i\right)^2 \left(\sum_{i=1}^n X_i^2\right)$$

and

$$\hat{\beta} = \left(\frac{1}{n}\sum_{i=1}^n X_i^2\right)\left(\sum_{i=1}^n X_i\right)^{-1} = \left(\sum_{i=1}^n X_i^2\right)\left(\sum_{i=1}^n X_i\right).$$

Q6

Solution. To avoid confusion, I will replace σ by α since it is an arbitrary variable. Then

$$\mathbb{E}X = \frac{1}{2\alpha}\int_{-\infty}^{\infty} x e^{-|x|/\alpha}\,\mathrm{d}x = 0$$

as the integrand is odd, and

$$\begin{aligned}\mathbb{E}X^2 &= \frac{1}{2\alpha}\int_{-\infty}^{\infty} x^2 e^{-|x|/\alpha}\,\mathrm{d}x = \frac{1}{\alpha}\int_0^{\infty} x^2 e^{-x/\alpha}\,\mathrm{d}x \\ &= \frac{1}{\alpha}\int_0^{\infty} (u\alpha)^2 e^{-u}\,\mathrm{d}u = \alpha^2\int_0^{\infty} u^2 e^{-u}\,\mathrm{d}x = \alpha^2\Gamma(3) = 2\alpha^2,\end{aligned}$$

so $\alpha = \sqrt{\mathbb{E}X^2/2}$. The method of moments estimator for α is therefore

$$\hat{\alpha} = \left(\frac{1}{2n}\sum_{i=1}^n X_i^2\right)^{1/2}.$$

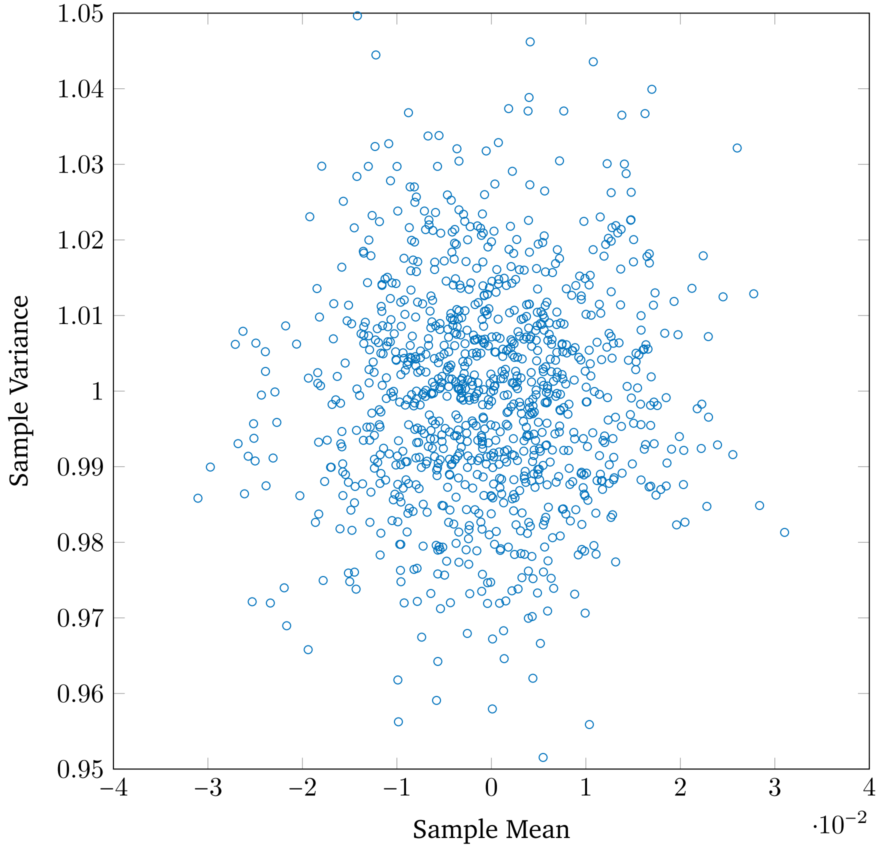
Q7

Solution. Code and scatterplot attached.

```
sample_mean = zeros(1,1000);
sample_var = zeros(1,1000);
for i = 1:1000
  sample = zeros(1,10000);
  for j = 1:10000
      sample(j) = normrnd(0,1);
  end
  sample_mean(i) = mean(sample);
  sample_var(i) = var(sample) / 9999 * 10000;
```

```
end

scatter(sample_mean, sample_var);
```



Q8

Solution. Code and output attached below. *Since* `tikz` *is not very good at handling large amount of samples, the plot corresponds to* $n = 10^6$.

```
Gaussian_array = randn(1,10^7); % 1*1mil Gaussian array
c = randn(2,10^7); % 2*1mil Gaussian array, first row Z1; second row Z2
chi_sq_array = sum(c.^2); % chi_sq with df2 = Z1^2 + Z2^2

quotient_array = Gaussian_array ./ (chi_sq_array /2).^0.5;

yyaxis left
histogram(quotient_array);
xlim([-10,10]);
ylim([0,2*10^5]);
hold on

yyaxis right
x = [-10:0.1:10];
stu = tpdf(x,2);
plot(x,stu,'r', 'LineWidth', 3);
```

```
xlim([-10,10]);

legend
```