# Math 408 Quiz 3 Sketch

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## Q1

*Proof.* Note that for each $x_i$,

$$f_\theta(x_i) = \begin{cases} 1/\theta & x_i \leqslant \theta \\ 0 & x_i > \theta. \end{cases}$$

Since $X_i$'s are i.i.d., for $x = (x_1, \dots, x_n)$ we have

$$f_\theta(x) = \prod_{i=1}^n f_\theta(x_i) = \theta^{-n} \chi_{[0,\theta]}(\max_{1 \leqslant i \leqslant n} x_i).$$

We see that $f_\theta(x)$ can be written entirely as a function $g_\theta$ depending on $X_{(n)}$ (and an identity function $h(x)$). Therefore by the factorization theorem $X_{(n)}$ is sufficient. Also, since $\chi_{[0,\theta]}(\max x_i)$ cannot be re-written as a function depending on $(x_1 + \dots + x_n)/n$ times a function depending on $x$, we see that the sample mean is not sufficient. □

## Q2

*Proof.* (1) Recall the identity $a^b = \exp(b \log a)$. Therefore the joint distribution is given by

$$\begin{aligned} f_{X_1,\dots,X_n}(x_1,\dots,x_n) &= \prod_{i=1}^n \frac{x_i^{\alpha-1}\exp(-x_i/\beta)}{\beta^\alpha \Gamma(\alpha)} \\ &= \frac{1}{\beta^{n\alpha}\Gamma(\alpha)^n} \cdot \prod_{i=1}^n x_i^{\alpha-1} \cdot \exp(-\sum_{i=1}^n x_i/\beta) \\ &= \frac{1}{\beta^{n\alpha}\Gamma(\alpha)^n} \cdot \prod_{i=1}^n \exp((\alpha-1)\log(x_i)) \cdot \exp(-\sum_{i=1}^n x_i/\beta) \\ &= \underbrace{\frac{1}{\beta^{n\alpha}\Gamma(\alpha)^n} \exp((\alpha-1)\sum_{i=1}^n \log(x_i))}_{g_\theta(Y)} \underbrace{\exp(-\sum_{i=1}^n x_i/\beta)}_{h(x)} \end{aligned} \tag{1}$$

which, by factorization theorem, shows that $\sum_{i=1}^n \log(X_i)$ is sufficient.

(2) Note that $\prod_{i=1}^n X_i = \exp(\sum_{i=1}^n \log(X_i))$, so

$$g_\theta(\sum_{i=1}^n \log(X_i)) = g_\theta(\log(\exp(\sum_{i=1}^n \log(X_i)))) = g_\theta(\log(\prod_{i=1}^n X_i)), \tag{2}$$

and modifying (1) using (2) shows that $\prod_{i=1}^n X_i$ is also sufficient.

(3) Since $X_i$'s are i.i.d., $\mathbb{E}(\prod_{i=1}^n X_i) = \prod_{i=1}^n \mathbb{E}X_i = \alpha^n \beta^n$.

(4) If $\beta = 1$ then $\mathbb{E}X = \alpha$. We know that $\overline{X} := n^{-1}\sum_{i=1}^n X_i$ is an unbiased and consistent estimator for $\alpha$, and we know that $\prod_{i=1}^n X_i$ is sufficient. Furthermore $\operatorname{var}(\overline{X})$ is finite. We define

$$\hat{\alpha}_n := \mathbb{E}_\theta[\overline{X}_n \mid \prod_{i=1}^n X_i].$$

It follows that $\hat{\alpha}_n$ is a function of $\prod_{i=1}^n X_i$ for all $n$:

$$\hat{\alpha}_n = g(Z) \text{ where } g(z) = \mathbb{E}_\theta[\overline{X}_n \mid \prod_{i=1}^n X_i = z].$$

By the total expectation theorem, $\mathbb{E}\hat{\alpha}_n = \mathbb{E}_\theta \overline{X}$ so $\hat{\alpha}$ is unbiased. Finally, by Rao-Blackwell,

$$\operatorname{var}(\hat{\alpha}_n) \leqslant \operatorname{var}(\overline{X}_n),$$

so $\lim_{n\to\infty} \operatorname{var}(\overline{X}_n) = 0$ implies $\lim_{n\to\infty} \operatorname{var}(\hat{\alpha}_n) = 0$. It remains to notice that this, along with $\mathbb{E}\hat{\alpha}_n = 0$, implies that $\hat{\alpha}_n$ is consistent: indeed, let $\epsilon > 0$ be given. By Chebyshev's inequality,

$$\mathbb{P}(|\hat{\alpha}_n - \alpha| > \epsilon) \leqslant \frac{\operatorname{var}(\hat{\alpha}_n)}{\epsilon^2} = \frac{\operatorname{var}(X)/n}{\epsilon^2} \to 0. \qquad \square$$

## Q3

*Proof.* For each $x_i$, the distribution function is given by

$$f_{X_i}(x_i) = \begin{cases} 1 & \theta - 1/2 \leqslant x \leqslant \theta + 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

Since the $X_i$'s are i.i.d. we have

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n) = \begin{cases} 1 & \text{if } x_i \in [\theta - 1/2, \theta + 1/2] \text{ for all } i \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the joint PDF can be written as

$$\chi_{[\theta-0.5,\infty)}(X_{(1)}) \cdot \chi_{(-\infty,\theta+0.5]}(X_{(n)})$$

which by the factorization theorem shows that the *pair* $(X_{(1)}, X_{(n)})$ is sufficient.
If we use $X_{(n)}$ alone, we lose information on how small the random sample can be, and likewise if we only use $X_{(1)}$ then we have no information on the maximum value of this random sample. $\square$

## Q4

*Proof.* Let $W := g(Z)$ where $g(z) := \mathbb{E}(Y \mid Z = z) = \int_{-\infty}^{\infty} y f_\theta(y \mid Z = z)\, \mathrm{d}y$. Since $Z$ is sufficient, $f_\theta(x \mid Z = z)$ does not depend on $\theta$. It follows that $f_\theta(y \mid Z = z)$, a function of $x$ only, has the same property. $\square$