MATH 525a Homework 1

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Problem 1

(a) Show that $[0,1]$ and $[0,1]^{\mathbb{N}}$ have the same cardinality.

(b) Show that $(0,1)^{\mathbb{N}}$ have the same cardinality as the sets in (a).

(c) Show that $\mathbb{R}^{\mathbb{N}}$ have the same cardinality as the sets in (a).

(d) Show that $\mathbb{R}$ has the same cardinality as the sets in (a).

Proof. I will reset the notation for each part, so the function f in (a) is totally unrelated to that in (b) and so on.

(a) According to the hint, we will make use of Schröder-Bernstein. Consider a mapping $f : [0,1] \to [0,1]^{\mathbb{N}}$ such that

$$0.x_1x_2... \mapsto (g : \mathbb{N} \to [0,1]) \text{ where } g(n) = x_n/10.$$

We further assume that $0.x_1x_2...$ does not end in a string of nines; if this is not the case, set all of them to 0 and increase the digit before the string by 1 (carrying allowed), and we have gotten rid of this nuisance, and every $x \in [0,1]$ has a unique decimal representation.

If $0.x_1x_2... \neq 0.y_1y_2...$, then for some n we have $x_n \neq y_n$, and so $[g(0.x_1x_2...)](n) \neq [g(0.y_1y_2...)](n)$. This means that the mapping f is injective and so $\text{card}([0,1]) \leqslant \text{card}([0,1]^{\mathbb{N}})$.

For the converse, let h be a function from $\mathbb{N}$ to $[0,1]$. For each $n \in \mathbb{N}$, $h(n)$ is a decimal with countably many digits; on the other hand, $\mathbb{N}$ contains countably many n's, so the union of all these digits is again countable, and there exists some method to arrange these digits and form a new decimal in $[0,1]$. The mapping from h to its corresponding number $\in [0,1]$ is clearly injective, so $\text{card}([0,1]^{\mathbb{N}}) \leqslant \text{card}([0,1])$. By Schröder-Bernstein, the two sets are equicardinal.

(b) By the same token we can easily show that $(0,1)$ and $(0,1)^{\mathbb{N}}$ are equicardinal. It remains to show that $\text{card}((0,1)) = \text{card}([0,1])$. Consider the bijection $f : (0,1) \to (0,1]$ by

$$f(x) = \begin{cases} 2x & x = 1/2^n, n \in \mathbb{N} \\ x & \text{otherwise.} \end{cases}$$

In other words, f maps the sequence $\{1/2, 1/4, 1/8, ...\}$ to $\{1, 1/2, 1/4, ...\}$ and does nothing to other numbers. Analogously we can show that there exists a bijection between $(0,1]$ to $[0,1]$. This proves the claim.

(c) First notice that the tangent function is a bijection between $(-\pi/2, \pi/2)$ and $\mathbb{R}$. Clearly we can scale the argument and obtain a bijection f between $(-1, 1)$ and $\mathbb{R}$.

To see that $\text{card}((0,1)^{\mathbb{N}}) = \text{card}(\mathbb{R}^{\mathbb{N}})$, consider the mapping $(g \in (0,1)^{\mathbb{N}}) \mapsto (f \circ g \in (\mathbb{R}^{\mathbb{N}}))$. If $g_1 \neq g_2$, then for some $n \in \mathbb{N}$ $g_1(n) \neq g_2(n)$, and so $(f \circ g_1)(n) \neq (f \circ g_2)(n)$, which implies $(f \circ g_1) \neq (f \circ g_2)$. The other direction's injectivity follows analogously since f is bijective. Therefore,

$$\text{card}(\mathbb{R}^{\mathbb{N}}) = \text{card}((0,1)^{\mathbb{N}}) = \text{card}([0,1]^{\mathbb{N}}).$$

(d) This follows from the previous three parts:

$$\text{card}(\mathbb{R}) = \text{card}((0,1)) = \text{card}([0,1]).$$

□

Problem 2

Show that $C([0,1])$ has the same cardinality as $\mathbb{R}$.

Proof. That $\text{card}(C([0,1])) \geqslant \text{card}(\mathbb{R})$ is clear: for each $x \in \mathbb{R}$ we just consider the constant function x. For the converse, consider $C(\mathbb{Q} \cap [0,1])$ instead: since $\mathbb{Q} \cap [0,1]$ is equicardinal to $\mathbb{N}$, we have

$$\text{card}(C(\mathbb{Q} \cap [0,1])) = \text{card}(\mathbb{R}^{\mathbb{N}}).$$

(Enumerate all rationals in $[0,1]$; at each rational, assign it the value that the function in $C(\mathbb{Q} \cap [0,1])$ takes.) It remains to notice that $C(\mathbb{Q} \cap [0,1])$ is equicardinal to $C([0,1])$: if $f, g \in C([0,1])$ agree on all rationals in $[0,1]$, then by density of $\mathbb{Q}$, for any $x \in [0,1]$ we can find a rational sequence $\{x_n\} \to x$. This implies

$$f(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = g(x),$$

so indeed $f \equiv g$. We have therefore proven the claim. □

Problem 3

Find a poset X which has a unique maximal element but no greatest type in the sense that there is no $x \in X$ for which $y \prec x$ for every $y \in X$.

Proof. Following the hint, consider $\mathcal{S} := \{(x,x) : x \in [0,1)\} \cup \{(1, 0.5)\}$, namely, the diagonal of the unit square excluding $(1,1)$ and with an extra point $(1, 0.5)$. We also define the partial order by

$$(x_1, y_1) \prec (x_2, y_2) \text{ if } x_1 \leqslant y_1 \text{ and } x_2 \leqslant y_2.$$

It is clear that $(1, 0.5)$ is a maximal element: if $(1, 0.5) \prec (x, y)$, then $x \geqslant 1$ and $y \geqslant 0.5$, but the only point satisfying $x \geqslant 1$ is $(1, 0.5)$ itself. On the other hand, $\mathcal{S}$ has no "greatest element": clearly $(1, 0.5)$ isn't because, say, $(0.75, 0.75)$ is not comparable with $(1, 0.5)$. It is also clear that no point on the *open* diagonal is maximal as there is no largest number in $[0, 1)$. Hence $\mathcal{S}$ does not have a "greatest" element.

Problem 5

One type of poset is a collection of subsets of some set Y, ordered by inclusion. In other words, the poset has form $(S, \preceq)$ where $S \subset \mathbb{P}(Y)$ and $A \preceq B$ means $A \subset B$. Show that every poset $(X, \leqslant)$ is isomorphic to some poset of this special type, that is, there exists Y and $S \subset \mathbb{P}(Y)$ and a bijection $f : X \to S$ such that $x \leqslant y$ if and only if $f(x) \preceq f(y)$.

Proof. Following the hint, let $I_x := \{z \in X : z \leqslant x\}$. From the Hausdorff Maximal Principle, let $X' \subset X$ be the largest totally ordered subset. Define $Y := X$ and

$$S := \{I_x : x \in X'\} \subset \mathbb{P}(Y).$$

We also define $f : X \to S$ by $f(x) = I_x$. We claim that $x \leqslant y$ if and only if $f(x) \preceq f(y)$.
To see $\Rightarrow$, let $\alpha \in f(x)$. It follows that $\alpha \leqslant x$ and therefore $\alpha \leqslant y$. Hence $\alpha \in f(y)$, so $f(x) \preceq f(y)$.
To see $\Leftarrow$, suppose $f(x) \npreceq f(y)$, that is, for some β we have $I_\beta \in f(x)$ but $I_\beta \notin f(y)$. Therefore, $\beta \leqslant x$ but $\beta \nleqslant y$. It follows from transitivity of $\leqslant$ that $x \nleqslant y$ (or we would have a contradiction). This proves the claim. ☐

Problem 6

Suppose X, Y, and Z are sets with $\text{card}(X) \leqslant \text{card}(Y)$, $\text{card}(Y) < \text{card}(Z)$. Prove that $\text{card}(X) < \text{card}(Z)$.

Proof. Since $\text{card}(X) \leqslant \text{card}(Y)$, there exists an injection $f : X \to Y$. Let $Y' \subset Y$ be the range of f. Since $\text{card}(Y) < \text{card}(Z)$, there exists an injection $g : Y \to Z$. It follows that $g|_{Y'}$ is also injective. Thus the composition $g|_{Y'} \circ f$ is an injection from X to Z and $\text{card}(X) \leqslant \text{card}(Y)$.
Next, since $\text{card}(Y) < \text{card}(Z)$, there cannot exist a surjection from Y to Z (or Schröder-Bernstein would imply that they have the same cardinality). Also note that, for the g above, any function $h : Z \to Y$ of form

$$h(z) = \begin{cases} y & \text{if } g(y) = z \text{ for some } y \\ \text{any } k \in Y & \text{if no } g(y) = z \end{cases}$$

is a surjection from Z to Y, so by Schröder-Bernstein there cannot be an injection from Z to Y. This further implies no injection exists between Z and X, and we recover the strict inequality $\text{card}(X) < \text{card}(Z)$. ☐

Problem 7

Let $Y \subset X$ where Y is at most countable and $X - Y$ is infinite. Show that there exists a bijection between $X - Y$ and X.

Proof. Since $X - Y$ is infinite, we are able to find a countably infinite subset $Z \subset X - Y$. WLOG assume Y and Z are disjoint. It is clear that there exists a bijection $f : Z \to Y \cup Z$. Now consider the function $g : (X - Y) \to X$ defined by

$$g(x) = \begin{cases} f(x) & x \in Z \\ x & x \in X - Y - Z. \end{cases}$$

This is a bijection and we are done. ☐

Problem 8

Suppose $P = (X, \leqslant)$ is a finite poset containing a smallest element x_0, that is, $x_0 \leqslant x$ for all $x \in X$. Suppose $f : X \to X$ is order-preserving, that is, $x \leqslant y \Rightarrow f(x) \leqslant f(y)$. Show that f has a fixed point, that is, an $x \in X$ such that $f(x) = x$.

Proof. Since x_0 is the smallest element, either $x_0 = f(x_0)$ or $x_0 < f(x_0)$. The former case already provides a fixed point. Since f is order-preserving, if $x_0 < f(x_0)$ we have

$$x_0 < f(x_0) \leqslant f^2(x_0) \leqslant f^3(x_0) \leqslant \dots$$

where f^n denotes the composition of n copies of f. This is an infinite sequence and a subset of X, yet X is finite. Therefore, there must exist some n such that $f^n(x) = f^{n+1}(x)$, and so $f^n(x) \in X$ is a fixed point of f. □