MATH 525a Homework 2

Qilin Ye

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Problem 1: (Folland §1.3)

Let $\mathfrak{M}$ be an infinite σ-algebra. Prove that

(1) $\mathfrak{M}$ contains an infinite sequence of disjoint sets, and

(2) $\text{card}(\mathfrak{M}) \geqslant \mathfrak{c}$.

Proof. (a) Since $\mathfrak{M}$ is infinite, we may construct a sequence of sets $\{E_i\}$ by setting $E_1 := \varnothing$ and choosing

$$E_i := \text{any set in } \mathfrak{M} - \{E_1, \dots, E_{i-1}\}.$$

Since $\mathfrak{M}$ is infinite whereas the power set of $\{E_1, \dots, E_n\}$ is finite, it follows that we can obtain a countable set $\{E_i\}_{i \geqslant 1}$. It is also clear that they are pairwise disjoint. This proves (a).

(b) We consider a function $f : \mathbb{P}(\mathbb{N}) \to \mathfrak{M}$ by

$$S \mapsto \bigcup_{i \in S} E_i.$$

Since $\mathfrak{M}$ is closed under countable unions, this map is well-defined; furthermore, since E_i's are disjoint, if $S_1 \neq S_2$ then $f(S_1) \neq f(S_2)$ so f is injective. It follows that

$$\text{card}(\mathfrak{M}) \geqslant \text{card}(\mathbb{N}) = \mathfrak{c}.$$ □

Problem 2: (Folland §1.4)

Prove that an algebra $\mathcal{A}$ is a σ-algebra if and only A is closed under countable increasing unions, i.e., if $\{E_i\} \subset \mathcal{A}$ and $E_1 \subset E_2 \subset \dots$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$.

Proof. Note that $\Rightarrow$ follows directly from the definition of σ-algebra.
For $\Longleftarrow$, let $\{S_i\}$ be a countable (countably infinite) collection of sets in $\mathcal{A}$. We define $E_n := \bigcup_{i=1}^{n} S_i$. It follows immediately that $\{E_n\}$ is an increasing sequence of sets whose union, by assumption, lies in $\mathcal{A}$. This completes the proof as

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} \bigcup_{i=1}^{n} S_i = \bigcup_{i=1}^{\infty} S_i.$$ □

Problem 3: (Folland §1.6)

Complete the proof of Theorem 1.9 (completion of measures).

Proof. To show that $\overline{\mu}$ is a complete measure, let A be such that $A \subset B$ and $\overline{\mu}(B) = 0$. By definition we know $B = E \cup F$ for some $E \in \mathfrak{M}$ and $F \subset N$ with $\mu(N) = 0$. Also, by definition we know $\mu(E) = \overline{\mu}(B) = 0$. Since

$$A = \varnothing \cup A$$

implies $A \in \overline{\mathfrak{M}}$ (indeed, $\varnothing \in \mathfrak{M}$ and $A \subset B$ where $\mu(B) = 0$), by monotonicity of $\overline{\mu}$ we have $\overline{\mu}(A) = 0$.
Next, to show that $\overline{\mu}$ is unique, suppose that $\overline{\mu}, \overline{\mu}'$ extend μ on $\mathfrak{M}$ to on $\overline{\mathfrak{M}}$. Let $A = E \cup F \in \overline{\mathfrak{M}}$ where $E \in \mathfrak{M}$ and $F \subset N \in \mathfrak{M}$ for some N satisfying $\mu(N) = 0$. Note that $E, E \cup N \in \mathfrak{M}$, so

$$\mu(E) = \overline{\mu}'(E) \leqslant \overline{\mu}'(A) \leqslant \overline{\mu}'(E \cup N) = \mu(E \cup N) \leqslant \mu(E) + \mu(N) = \mu(E),$$

so the inequalities must all be ='s, and so $\overline{\mu}'(A) = \mu(E) = \overline{\mu}(E)$. This proves uniqueness. □

Problem 4: (Folland §1.8)

Let $(X, \mathfrak{M}, \mu)$ be a measure space and $\{E_i\}_{i=1}^{\infty} \subset \mathfrak{M}$,. Prove that $\mu(\liminf E_i) \leqslant \liminf \mu(E_i)$. Also prove that if $\mu(\bigcup_{i=1}^{\infty} E_i) < \infty$ then $\mu(\limsup E_i) \geqslant \limsup \mu(E_i)$.

Proof. Following the hint, we consider $\{F_k\}$ and $\{G_k\}$ where $F_k := \bigcap_{n=k}^{\infty} E_n$ and $G_k := \bigcup_{n=k}^{\infty} E_n$.
On one hand, $F_1 \subset F_2 \subset \ldots$ so continuity from below implies

$$\begin{aligned}\mu(\liminf E_i) = \mu(\bigcup_{n\geqslant 1} \bigcap_{i \geqslant n} E_i) = \mu(\bigcup_{n \geqslant 1} F_n) &= \lim_{n\to\infty} \mu(F_n) \\ &\leqslant \lim_{n\to\infty} (\inf_{i\geqslant n} \mu(E_i)) = \liminf \mu(E_i).\end{aligned}$$

On the other hand, $G_1 \supset G_2 \supset \ldots$. If we also assume that $\mu(G_1) < \infty$, then continuity from above implies

$$\begin{aligned}\mu(\limsup E_i) = \mu(\bigcap_{n\geqslant 1} \bigcup_{i \geqslant n} E_i) = \mu(\bigcap_{n \geqslant 1} G_n) &= \lim_{n\to\infty} \mu(G_n) \\ &\geqslant \lim_{n\to\infty} (\sup_{i\geqslant n} \mu(E_i)) = \limsup \mu(E_i).\end{aligned}$$

□

Problem 5: (Folland §1.12)

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Prove that:

(a) If $E, F \in \mathfrak{M}$ and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$;

(b) If we say $E \sim F$ if $\mu(E\Delta F) = 0$, then $\sim$ defines an equivalence relation on $\mathfrak{M}$; and

(c) For $E, F \in \mathfrak{M}$, define $\rho(E, F) := \mu(E \Delta F)$. Then $\rho(E, G) \leqslant \rho(E, F) + \rho(F, G)$ and hence ρ defines a metric on the space $\mathfrak{M}/\sim$.

Proof. (a) If $\mu(E\Delta F) = \mu(E-F) + \mu(F-E) = 0$ then clearly $\mu(E-F) = \mu(F-E)$. Hence

$$\mu(E) = \mu(E-F) \cup \mu(E\cap F) = \mu(F-E) \cup \mu(E \cap F) = \mu(F).$$

(b) Reflexivity and symmetry of $\sim$ are clearly trivial. For transitivity, assume that $E \sim F$ and $F \sim G$; that is,

$$\mu(E\Delta F) = \mu(E \cap F^c) + \mu(F^c \cap E) = \mu(F\Delta G) = \mu(F\cap G^c) + \mu(G^c \cap F) = 0.$$

(Of course, notice that $E\cap F^c$ is just another way of writing $E-F$, but this will make things more convenient below.) Then,

$$\begin{aligned}\mu(E\Delta G) &= \mu(E\cap G^c) + \mu(E^c \cap G)\\ &= \mu(E\cap F\cap G^c) + \mu(E\cap F^c \cap G^c) + \mu(E^c\cap F\cap G) + \mu(E^c \cap F^c \cap G)\\ &\leqslant \mu(F\cap G^c) + \mu(F^c\cap G^c) + \mu(E^c \cap F) + \mu(E^c\cap F^c) = 0.\end{aligned}$$

(c) The computation is identical to those in (b), with the last $= 0$ replaced by $= \mu(E\Delta F) + \mu(F\Delta G) = \rho(E,F) + \rho(F,G)$. □

Problem 6: (I)

Suppose $\{\mu_n\}$ is a sequence of finite measures on $(X, \mathfrak{M})$ and $\mu_n \uparrow \mu$ on $\mathfrak{M}$ for some set function μ. Show that μ is a measure.

Proof. That $\mu(\varnothing) = 0$ is clear so it remains to show countable additivity for disjoint sets. To this end, let $\{E_i\}$ be a countable collection of disjoint sets in $\mathfrak{M}$. Then,

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \lim_{n\to\infty} \mu_n\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_n(E_i) \leqslant \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

On the other hand,

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \lim_{n\to\infty} \mu_n\Big(\bigcup_{i=1}^{\infty} E_i\Big) \geqslant \lim_{n\to\infty} \mu_n\Big(\bigcup_{i=1}^{k} E_i\Big) = \lim_{n\to\infty} \sum_{i=1}^{k} \mu_n(E_i) = \sum_{i=1}^{k} \lim_{n\to\infty} \mu_n(E_i) = \sum_{i=1}^{k} \mu(E_i).$$

Taking the supremum over k, we obtain $\mu(\bigcup_{i=1}^{\infty} E_i) \geqslant \sup_{k\in\mathbb{N}} \sum_{i=1}^{k} \mu(E_i) = \sum_{i=1}^{\infty} \mu(E_i)$. □

Problem 7: (II)

Let $\mathcal{B}_{\mathbb{R}}$ be the Borel sets in $\mathbb{R}$, so by definition $\mathcal{B}_{\mathbb{R}}$ is generated by the collection $\mathcal{E}$ of all open intervals. Define the measure

$$\mu(E) = \begin{cases} 0 & \text{if } E = \varnothing, \\ \infty & \text{if } E \neq \varnothing. \end{cases}$$

Find another measure ν such that $\mu = \nu$ on the generators $\mathcal{E}$ but μ and ν do not agree on all of $\mathcal{B}_{\mathbb{R}}$.

Solution. Consider ν as the counting measure — any open interval is uncountable so $\nu(E) = \infty$ when $E \neq \varnothing$. Clearly, $\nu(\varnothing) = \varnothing$. However, $\mu(\{0\}) = \infty \neq 1 = \nu(\{0\})$, so μ and ν are not identical.

Problem 8: (III)

Let $\mathcal{A}$ be a collection of pairwise disjoint subsets of a σ-finite measure space, and suppose that each set in $\mathcal{A}$ has strictly positive measure. Show that $\mathcal{A}$ is at most countable.

Proof. By σ-finiteness (of our ambient space X), there exist a contable collection $\{E_i\}$ such that $X = \bigcup_{i=1}^{\infty} E_i$ and $\mu(E_i) < \infty$ for all i.
Now, for each $A \in \mathcal{A}$, since $\mu(A) > 0$, we know that $\mu(A \cap E_i) > 0$ for some i. Since there are only countably many E_i's but uncountably many A's, by pigeon-hole there exists some E_j satisfying $\mu(A \cap E_j) > 0$ for uncountably many A's in $\mathcal{A}$. For convenience let S denote the collection of such A's. Now we define

$$S_k := \{A \in \mathcal{A} \mid \mu(A \cap E_j) > 1/k\}.$$

It follows that there are countably many S_n's whose union is S. Once again, by pigeon-hole, since S is uncountable but each S_k is countable, for some n we have an uncountable S_n. But then

$$\mu(E_j) \geqslant \mu(S_k) = \sum_{A \in S_k} \mu(A \cap E_j) > \sum_{A \in S_k} 1/k = \infty,$$

contradicting our assumption that $\mu(E_j) < \infty$. Hence $\mathcal{A}$ is at most countable. □

Problem 9: (IV)

Suppose $\mathcal{E}, \mathcal{F}$ are subsets of $\mathbb{P}(X)$ with $\mathcal{E} \subset \mathcal{F} \subset \sigma(\mathcal{E})$. Show that $\sigma(\mathcal{F}) = \sigma(\mathcal{E})$.

Proof. Since $\sigma(\mathcal{E})$ is the smallest σ-algebra containing $\mathcal{E}$ and $\sigma(\mathcal{F})$ is also a σ-algebra, we have $\sigma(\mathcal{F}) \supset \sigma(\mathcal{E})$. On the other hand, since $\sigma(\mathcal{F})$ is the smallest σ-algebra containing $\mathcal{F}$ and $\sigma(\mathcal{E})$ is a σ-algebra, we also have $\sigma(\mathcal{F}) \subset \sigma(\mathcal{E})$. Therefore $\sigma(\mathcal{E}) = \sigma(\mathcal{F})$. □

Problem 10: (V)

Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $\{E_i\}$ be a sequence of sets in $\mathfrak{M}$ such that each E_i intersects at most one other set in the sequence. Show that

$$\sum_{i=1}^{\infty} \mu(E_i) \leqslant 2\mu(\bigcup_{i=1}^{\infty} E_i).$$

Proof. For each E_i, define F_i to be the part where E_i intersects with some other E_j and G_i to be $E_i - F_i$. It is immediately clear that $\{G_i\}_{i=1}^{\infty}$ is a collection of pairwise disjoint sets. Since one set intersects with at most one other set, each nontrivial element (meaning not $\varnothing$) in $\{F_i\}_{i=1}^{\infty}$ is counted exactly twice. Therefore there exists a partition $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\{F_i\}$ such that the sets in $\mathcal{F}_1$ are pairwise disjoint and likewise for those in $\mathcal{F}_2$.

Furthermore, sets in $\mathcal{F}_1 \cup \{G_i\}$ are also pairwise disjoint and their union is precisely $\bigcup_{i=1}^{\infty} E_i$. Therefore,

$$\begin{aligned}\sum_{i=1}^{\infty} \mu(E_i) &= \sum_{i=1}^{\infty} \mu(F_i) + \sum_{i=1}^{\infty} \mu(G_i) = \sum_{\mathcal{F}_1} \mu(F_i) + \sum_{i=1}^{\infty} \mu(G_i) + \sum_{\mathcal{F}_2} \mu(F_i) \\ &= \sum_{\mathcal{F}_1 \cup \{G_i\}} \mu(\,\cdot\,) + \sum_{\mathcal{F}_2} \mu(F_i) = \mu(\bigcup_{i=1}^{\infty} E_i) + \mu(\bigcup_{\mathcal{F}_2} E_i) \\ &\leqslant \mu(\bigcup_{i=1}^{\infty} E_i) + \mu(\bigcup_{i=1}^{\infty} E_i) = 2\mu(\bigcup_{i=1}^{\infty} E_i).\end{aligned}$$

□