MATH 525a Homework 3

Qilin Ye

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Problem 1: (Folland, 1.26)

Prove that if $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$ then for every $\epsilon > 0$ there is a set A that is a finite union of open intervals such that $\mu(E \Delta A) < \infty$. (*Hint: use Theorem 1.18.*)

Proof. By Theorem 1.18, given E, there exist $U \supset E$ open and $K \subset E$ compact such that

$$\mu(E - K) < \frac{\epsilon}{2} \qquad \text{and} \qquad \mu(U - E) < \frac{\epsilon}{2}.$$

Next, since U is open, it can be written as a countable union of open intervals, i.e., $E = \bigcup_{i=1}^{\infty} I_i$. (The proof is given in Folland's chapter 0.) Since K is compact, from $K \subset \bigcup_{i=1}^{\infty} I_i$ we know that we can extract a finite subcover $\{I_1, \ldots, I_n\}$. If we define their finite union to be A, then

$$\mu(E \Delta A) = \mu(E - A) + \mu(A - E) \leqslant \mu(E - K) + \mu(U - E) = \epsilon.$$

□

Problem 2: (Folland, 1.30)

If $E \in \mathcal{L}$ and $m(E) > 0$, prove that for any $\alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$.

Proof. Suppose not; that is, there exists $\alpha < 1$ such that $m(E \cap I) \leqslant \alpha m(I)$ for all open intervals I. Consider a collection $\{I_i\}_{i=1}^{\infty}$ that covers E (this is always possible since even $\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n)$), from which we have

$$m(E) = m(\bigcup_{i=1}^{\infty} E \cap I_i) \leqslant \sum_{i=1}^{\infty} m(E \cap I_i) \leqslant \alpha \sum_{i=1}^{\infty} m(I_i).$$

Taking the infimum over all possible covers of E using open intervals, we obtain

$$m(E) \leqslant \alpha \inf\{\sum_{i=1}^{\infty} m(I_i) : \bigcup_{i=1}^{\infty} I_i \supset E\} = \alpha m(E) < m(E),$$

clearly a contradiction. This proves the claim. □

Problem 3: (I)

Suppose μ^* is an outer measure which is finitely additive. Show that μ^* is actually a measure.

Proof. We only need to verify countable additivity of disjoint sets, of which the subadditivity direction also follows from definition.

Now let $\{E_i\}$ be a countable collection of disjoint sets. By finite additivity

$$\mu^*(\bigcup_{i=1}^{\infty} E_i) \geqslant \mu^*(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mu^*(E_n)$$

for all n, so letting $n \to \infty$ we obtain $\geqslant$ as well, proving the claim. □

Problem 4: (II)

Suppose f is Lipschitz and $E \subset \mathbb{R}$ has Lebesgue measure 0. Show that $m(\{f(x) : x \in E\}) = 0$ also.

Proof. Since $m(E) = \inf\{\sum_{i=1}^{\infty} m(b_i - a_i) : \bigcup_{i=1}^{\infty}(a_i, b_i) \supset E\}$, there exists a covering $\mathcal{K} := \{(a_i, b_i)\}$ that is "ϵ-optimal", i.e.,

$$\sum_{i=1}^{\infty} m(b_i - a_i) < m(E) + \epsilon = \epsilon.$$

Since f is Lipschitz, each (a_i, b_i) can be mapped to an interval of length at most $L(b_i - a_i)$, where L is the corresponding Lipschitz constant. Since

$$\{f(x) : x \in E\} \subset \{f(a_i, b_i) : (a_i, b_i) \in \mathcal{K}\}$$

we have

$$m(\{f(x) : x \in E\}) \leqslant \sum_{i=1}^{\infty} m(f(a_i, b_i)) \leqslant L \sum_{i=1}^{\infty} m(b_i - a_i) < L\epsilon.$$

Since ϵ is arbitrary we see that $\{f(x) : x \in E\}$ indeed has Lebesgue measure 0. □

Problem 5: (III)

Let $(X, \mathfrak{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $\mathcal{A} \subset \mathfrak{M}$ be an algebra (not necessarily a σ-algebra!). A set $E \in \mathfrak{M}$ is called **approximable from inside by $\mathcal{A}$** if for every $\epsilon > 0$ there exists $A \in \mathcal{A}$ with $A \subset E, \mu(E - A) < \epsilon$.

Show that $\mathcal{C} := \{E \in \mathfrak{M} : E$ is approximable from inside by $\mathcal{A}\}$ is closed under countable unions.

Proof. Let $\{E_n\}$ be a countable collection of sets in $\mathcal{C}$, and let $\{A_n\}$ be the corresponding sets in $\mathcal{A}$ that can approximate E from inside with error $< \epsilon 2^{-n}$, i.e., $\mu(E_n - A_n) < \epsilon 2^{-n}$. We now consider $\bigcup_{n=1}^{\infty} E_n - \bigcup_{n=1}^{k} A_n$ (since $\mathcal{A}$ is not a σ-algebra, we should instead consider finite unions of A_n's):

$$\begin{aligned}
\mu(\bigcup_{n=1}^{\infty} E_n - \bigcup_{n=1}^{k} A_n) &= \mu(\bigcup_{n=1}^{\infty} E_n - \bigcup_{n=1}^{k} E_n) + \mu(\bigcup_{n=1}^{k} E_n - \bigcup_{n=1}^{k} A_n) \\
&\leqslant \mu(\bigcup_{n=1}^{\infty} E_n - \bigcup_{n=1}^{k} E_n) + \mu(\bigcup_{n=1}^{k} (E_n - A_n)) \\
&\leqslant \mu(\bigcup_{n=1}^{\infty} E_n - \bigcup_{n=1}^{k} E_n) + \sum_{n=1}^{k} \mu(E_n - A_n) < \mu(\bigcup_{n=1}^{\infty} E_n - \bigcup_{n=1}^{k} E_n) + \epsilon.
\end{aligned}$$

Note that the first term can be re-written as $\bigcup_{n=k+1}^{\infty} E_n$, and as k increases, this forms a nested decreasing sequence.

With the additional assumption that $\mu(X) < \infty$, we may use continuity from below and see that

$$\lim_{k\to\infty} \mu(\bigcup_{n=1}^{\infty} E_n - \bigcup_{n=1}^{k} E_n) = \mu(\varnothing) = 0,$$

so there exists a sufficiently large K such that $\mu(\bigcup_{n=1}^{\infty} E_n - \bigcup_{n=1}^{K} E_n) < \epsilon$. For such K,

$$\mu(\bigcup_{n=1}^{\infty} E_n - \bigcup_{n=1}^{k} A_n) < 2\epsilon.$$

Since ϵ is arbitrary, we have done showing that $\mathcal{C}$ is closed under countable unions. □

Problem 6: (IV)

Let X be uncountable. For $A \subset X$, define

$$\mu^*(A) := \begin{cases} 1 & \text{if } A \text{ is uncountable} \\ 0 & \text{if } A \text{ is countable.} \end{cases}$$

(a) If A is a countable subset of X, show that A is μ^*-measurable.

(b) If A and A^c are both uncountable, show that A is not μ^*-measurable.

Proof. (a) Let A be countable so that $\mu^*(A) = 0$. If we take any $E \in X$ and consider $A \cap E$ and $A \cap E^c$, it is obvious that both intersections need to be countable, so

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \qquad \text{for all } E \in X,$$

which by definition means A is μ^*-measurable.

(b) A is not μ^*-measurable because

$$\mu^*(X) = 1 < 2 = \mu^*(X \cap A) + \mu^*(X \cap A^c).$$

□

Problem 7: (V)

For X a separable metric space, and μ a measure on $\mathcal{B}_X$, the **support** of μ is the smallest closed set F with $\mu(F^c) = 0$.

(a) Suppose $F \subset \mathbb{R}$ is closed. Show that there is a finite measure ν on $\mathcal{B}_{\mathbb{R}}$ whose support is the set F.

(b) Suppose the measure ν on $\mathcal{B}_{\mathbb{R}}$ is finite on bounded sets, and let F be its distribution function. Show that $x \in \operatorname{supp}(\nu)$ if and only if $F(x') < F(x'')$ for all $x' < x < x''$.

Proof. (*) Following the hint, we first show that F has a countable dense subset. If F is bounded, then by

Heine-Borel it is compact; thus given $n \in \mathbb{N}$, the collection

$$\bigcup_{x \in F} B(x, 1/n) \supset F$$

admits a finite cover. Call this finite collection of these radii S_n. It follows that $\bigcup_{n=1}^{\infty} S_n$ is a countable set that is dense in F.

On the other hand, if F is unbounded, we first construct a countable dense subset for $F \cap [-k, k]$ for each $k \in \mathbb{N}$. Taking the countable union over $k \in \mathbb{N}$, we obtain another countable dense subset that is dense in $\bigcup_{k=1}^{\infty} [-k, k] \supset F$. Therefore F also admits a countable dense subset.

(a) Let A be a countable dense subset of F and let $\{a_n\}_{n \geqslant 1}$ be an enumeration of A. We define

$$\mu(a_n) := 2^{-n} \qquad \text{and} \qquad \mu \equiv 0 \text{ everywhere else.}$$

Then μ is a finite measure with $\mu(\mathbb{R}) = 1$ and the support of $\mu = \overline{A} = F$.

(b) If $x \notin \text{supp}(\nu)$, then there exists $\epsilon > 0$ such that $\nu \equiv 0$ on $(x-\epsilon, x+\epsilon)$ (because by definition its distribution function is 0 on some sufficiently small interval). Then if we pick $x' \in (x - \epsilon, x)$ and $x'' \in (x, x + \epsilon)$ we see that $F(x') = F(x'')$ whereas $x' < x < x''$. This proves $\Leftarrow$.

For $\Rightarrow$, let $x \in \text{supp}(\nu)$ and let $x' < x < x''$. Let ϵ be such that $x' < x - \epsilon < x + \epsilon < x''$. Then by definition of support, there exists some $t \in (x - \epsilon, x + \epsilon)$ such that $F(t) > 0$ (where F is the distribution). This implies $\mu((x - \epsilon, x + \epsilon)) > 0$, so

$$F(x'') - F(x') = \mu((x', x'')) \leqslant \mu(x - \epsilon, x + \epsilon) > 0.$$

□

Problem 8: (VI)

(a) For $x \in \mathbb{R}$ and $E \subset \mathbb{R}$. Show that the Lebesgue measure m on $\mathbb{R}$ is translation-invariant: $\mu(E) = \mu(x+E)$ for all $x \in \mathbb{R}$ and all Lebesgue-measurable E.

(b) From (a) it follows readily that Lebesgue measure on $[0, 1]$ is translation-invariant "mod 1" in the sense of the nonmeasurable sets given in the beginning of the course. Show that if A is that set and $E \subset A$ is Lebesgue measurable then $\mu(E) = 0$.

Proof. (a) Let $\bigcup_{i=1}^{\infty}(a_i + x, b_i + x)$ be any open cover of $x + E$. Then,

$$\sum_{i=1}^{\infty} \mu(a_i + x, b_i + x) = \sum_{i=1}^{\infty} \mu(a_i + b_i) \geqslant \inf\{\sum_{i=1}^{\infty} \mu(a_i, b_i) : \bigcup_{i=1}^{\infty}(a_i, b_i) \supset E\},$$

so taking the infimum over all open covers of $x + E$ gives $\mu(x + E) \geqslant \mu(E)$. Likewise, E itself is a translate of $x + E$, so $\mu(E) \geqslant \mu(x + E)$. Therefore $\mu(E) = \mu(x + E)$, i.e., μ is translation-invariant.

(b) We define $E_q := E + q$ and consider $\{E_q : q \in [-1, 1] \cap \mathbb{Q}\}$. It follows that the E_q's are pairwise disjoint and that the union is a subset of $[-1, 2]$. Let $\{E_1, E_2, \ldots\}$ be an enumeration of that collection. By countable additivity,

$$\mu(\bigcup E_q) = \sum_{i=1}^{\infty} \mu(E_q) = \sum_{i=1}^{\infty} \mu(E) \leqslant \mu([-1,2]) = 3,$$

and we easily see that a contradiction arises unless $\mu(E) = 0$. □

Problem 9: (VII)

(a) Let X be an infinite set, and for $E \subset X$ define

$$\mu^*(E) := \begin{cases} 0 & \text{if } E = \varnothing, \\ 1 & \text{if } E \text{ is nonempty and finite} \\ \infty & \text{if } E \text{ is infinite.} \end{cases}$$

Show μ^* is an outer measure.

(b) Prove or disprove that every outer measure ν^* is continuous from below, i.e., if $E_1 \subset E_2 \subset \dots$ then

$$\nu^*(\bigcup_{n=1}^{\infty} E_n) = \lim_{n \to \infty} \nu^*(E_n).$$

Proof. (a) It is clear that $\nu^*(\varnothing) = 0$ and that $\nu^*(A) \leqslant \nu^*(B)$ if $A \subset B$, so it remains to verify countable subadditivity. Suppose for a countable collection $\{E_n\}$ of sets we have $\mu^*(\bigcup_{n=1}^{\infty} E_n) > \sum_{n=1}^{\infty} \mu^*(E_n)$.

The LHS of the inequality can be either 1 or ∞. If it is 1 then the RHS is 0, meaning that each $E_n = \varnothing$, but then their union is also $\varnothing$, contradicting LHS $= 1$. If it is ∞ then clearly some $E_n \neq \varnothing$, so the RHS is forced to be 1. This means that precisely one E_n is nonempty and furthermore it needs to be finite. Then $E_n \cup \varnothing \cup \varnothing \cup \dots = E_n$, so the LHS is $\mu^*(E_n) = 1$, again contradicting our assumption that LHS $= \infty$. Therefore $\mu^*(\bigcup_{n=1}^{\infty} E_n) > \sum_{n=1}^{\infty} \mu^*(E_n)$ can never happen and we are done showing that μ^* is an outer measure.

(b) The claim is false. Consider $E_n := \{1, \dots, n\}$. Then $E_1 \subset E_2 \subset \dots$ and let $\nu^* := \mu^*$ as defined in (a). Then

$$\nu^*(\bigcup_{n=1}^{\infty} E_n) = \nu^*(\mathbb{N}) = \infty \neq 1 = \lim_{n \to \infty} 1 = \lim_{n \to \infty} \nu^*(E_n).$$ □