MATH 525a Homework 4

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Problem: (Folland 2.3)

If $\{f_n\}$ is a sequence of measurable functions on X, prove $\{x : \lim f_n(x) \text{ exists}\}$ is a measurable set.

Proof. Note that $[\lim f_n(x) \text{ exists}] \Leftrightarrow [\limsup f_n(x) - \liminf f_n(x) = 0]$. By proposition 2.7, $\limsup f_n$ and $\liminf f_n$ are both measurable, and by proposition 2.6, so is $g := \limsup f_n - \liminf f_n$. It remains to notice that

$$\{x : \lim f_n \text{ exists}\} = g^{-1}(\{0\})$$

so this is indeed a measurable set. $\square$

Problem: (Folland 2.13)

Suppose $\{f_n\} \subset L^+$, $f_n \to f$ pointwise, and $\int f = \lim \int f_n < \infty$. Show that $\int_E f = \lim \int_E f_n$ for all $E \in \mathfrak{M}$. However this need not to be true if $\int f = \lim \int f_n = \infty$.

Proof. Writing f as $\liminf f_n$, for any $E \in \mathfrak{M}$ we have

$$\begin{aligned}
\int_E f = \int_E \liminf f_n \leqslant \liminf \int_E f_n \leqslant \limsup \int_E f_n & & \text{(Fatou)} \\
= \limsup\Big(\int f_n - \int_{E^c} f_n\Big) & & \\
\overset{*}{=} \lim \int f_n - \liminf \int_{E^c} f_n = \int f - \liminf \int_{E^c} f_n & & \\
\leqslant \int f - \int_{E^c} \liminf f_n = \int f - \int_{E^c} f = \int_E f. & & \text{(Fatou again)}
\end{aligned}$$

The starred equation is because if $x_n \to x$ then $\limsup(x_n - y_n) = \lim x_n + \limsup(-y_n) = \lim x_n - \liminf y_n$ (a standard fact from 425a; proof attached below). Therefore all inequalities must attain equality and in particular

$$\int_E f = \liminf \int_E f_n = \limsup \int_E f_n = \lim \int_E f_n.$$

Proof of subclaim. We use Rudin's [?] definition: $\limsup x_n := \sup E$ where E is the set of subsequential limits of $\{x_n\}$ (and likewise $\liminf = \inf E$). Let E_y be the set corresponding to y_n. Since $x_n \to x$, the set corresponding to $\{x_n\}$ is simply $\{x\}$. Since x_n converges to x, $\{x_{n_k} + y_{n_k}\}$ converges to p if and only if the corresponding $\{y_{n_k}\}$ converges to $p - x$, the set of subsequential limits of $\{x_n + y_n\}$ is simply $x + E_y$. Thus

$$\sup(x + E_y) = x + \sup(E_y) \implies \limsup(x_n + y_n) = \lim x_n + \limsup y_n.$$

This, along with $\limsup(-y_n) = -\liminf(y_n)$, gives the subclaim. END OF PROOF OF SUBCLAIM

If we allow $\int f = \lim \int f_n := \infty$, consider $f_n := n\chi_{(0,1/n)} + \chi_{[1,\infty)}$ and $f := \chi_{[1,\infty)}$, both defined on $(0,\infty)$. Let $E = (0,1)$. Then $\int_E f_n = 1$ for all n whereas $\int_E f = 0$. □

Problem: (Folland 2.14)

If $f \in L^+$, let $\lambda(E) := \int_E f \, \mathrm{d}\mu$ for $E \subset \mathfrak{M}$. Show that λ is a measure on $\mathfrak{M}$ and for any $g \in L^+$, $\int g \, \mathrm{d}\lambda = \int fg \, \mathrm{d}\mu$. (Hint: first suppose that g is simple.)

Proof. It is clear that $\lambda(\varnothing) = 0$, so it remains to check countable additivity. First, if E_1, E_2 are disjoint then

$$\lambda(E_1 \cup E_2) = \int_{E_1 \cup E_2} f \, \mathrm{d}\mu = \int f\chi_{E_1 \cup E_2} \, \mathrm{d}\mu = \int f\chi_{E_1} \, \mathrm{d}\mu + \int f\chi_{E_2} \, \mathrm{d}\mu = \lambda(E_1) + \lambda(E_2),$$

so induction shows finite additivity. Now let $\{E_i\}_{i=1}^\infty \subset \mathfrak{M}$ be a countable collection of disjoint sets. By MCT

$$\lambda(\bigcup_{i=1}^\infty E_i) = \int_{\bigcup_{i=1}^\infty E_i} f \, \mathrm{d}\mu = \int f\chi_{\bigcup_{i=1}^\infty E_i} \, \mathrm{d}\mu = \lim_{n\to\infty} \int f\chi_{\bigcup_{i=1}^n E_i} \, \mathrm{d}\mu = \lim_{n\to\infty} \lambda(\bigcup_{i=1}^n E_i) = \sum_{i=1}^\infty \lambda(E_i).$$

This shows that λ is a measure.

For the second part, first assume g is simple. Let $\sum_{i=1}^n c_i\chi_{E_i}$ be a standard representation. Then

$$\int g \, \mathrm{d}\lambda = \sum_{i=1}^n c_i\lambda(E_i) = \sum_{i=1}^n c_i \int_{E_i} f \, \mathrm{d}\mu = \sum_{i=1}^n c_i \int f\chi_{E_i} \, \mathrm{d}\mu = \int \sum_{i=1}^n c_i f\chi_{E_i} \, \mathrm{d}\mu = \int fg \, \mathrm{d}\mu.$$

For general $g \in L^+$, let $\{\varphi_j\}$ be a sequence of simple functions with $\varphi_n \uparrow g$. By using MCT twice,

$$\int g \, \mathrm{d}\lambda = \int \lim_{j\to\infty} \varphi_j \, \mathrm{d}\lambda = \lim_{j\to\infty} \int \varphi_j \, \mathrm{d}\lambda = \lim_{j\to\infty} \int f\varphi_j \, \mathrm{d}\mu = \int \lim_{j\to\infty} f\varphi_j \, \mathrm{d}\mu = \int fg \, \mathrm{d}\mu.$$ □

Problem: (II)

Suppose f is measurable from $(X, \mathfrak{M})$ to $(\mathbb{R}, \mathcal{B}_\mathbb{R})$, and let $\mathcal{F} \subset \mathfrak{M}$ be the σ-algebra $\{f^{-1}(E) : E \in \mathcal{B}_\mathbb{R}\}$. Suppose g is measurable $(X, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}_\mathbb{R})$. Show that $g = h \circ f$ for some $h : \mathbb{R} \to \mathbb{R}$.

Proof. First we claim that if there does not exist such g, then for some $a, b \in X$, we have $f(a) = f(b)$ but $g(a) \neq g(b)$. Indeed, assuming the negation of the second statement, i.e., for all $a, b \in X$, if $f(a) = f(b)$ then $g(a) = g(b)$, the map $\mathbb{R} \to \mathbb{R}$ defined by $f(x) \mapsto g(x)$ is well-defined.

Now for contradiction, assume no g exists; we find $a, b \in X$ with the property above. Since g is measurable and

$g(a)$ is a singleton in $\mathbb{R}$, we see $S := g^{-1}(g(a)) \in \mathcal{F}$. □

Problem: (III)

Let $(X, \mathfrak{M}, \mu)$ be a measure space with μ finite, and let $f : X \to [0, \infty)$ be a measurable function. Show there exists an increasing $g : [0, \infty) \to [0, \infty)$ for which $\lim_{x\to\infty} g(x) = \infty$ but $g \circ f$ integrable.

Proof. We first prove the hint:

Lemma. If $\{a_n\} \geqslant 0$ and $\sum_{n=1}^{\infty} a_n < \infty$, then there exists $\{b_n\}$ with $b_n \to \infty$ but $\sum_{n=1}^{\infty} a_n b_n < \infty$.

Proof of lemma. Since $\sum_{n=1}^{\infty} a_n$ converges, by Cauchy convergence criterion, every $\epsilon_k := 2^{-k}$ corresponds to a n_k where $\sum_{n \geqslant n_k} a_n < 2^{-k}$. We define $b_n = 1$ for $n < n_1$ to get rid of the early large terms. Then define $b_n = k$ for $n_k \leqslant n < n_{k+1}$. On one hand, b_n clearly tends to ∞; on the other hand,

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{n_1-1} a_n + \sum_{k=1}^{\infty} \sum_{n=n_k}^{n_{k+1}-1} a_n b_n < \sum_{n=1}^{n_1-1} a_n + \sum_{k=1}^{\infty} \frac{k}{2^k} < \infty.$$

END OF PROOF OF LEMMA

Now for the main proof, define $E_n := f^{-1}([n-1, n))$. By construction $X = \bigcup_{n=1}^{\infty} E_n$ and E_n's are disjoint, so $\sum_{n=1}^{\infty} \mu(E_n)$ converges. We construct a sequence of real numbers $\{g_n\}$ according to the lemma (i.e., $g_n \to \infty$ with $\sum_{n=1}^{\infty} \mu(E_n) g_n < \infty$) and define a function such that $g(0) = 0$ and $g\big|_{(n-1,n]} \equiv g_n$. Clearly g is stepwise increasing. We claim that g has the desired property:

$$\begin{aligned}\int_X g \circ f \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_{E_n} g \circ f \, \mathrm{d}\mu &\leqslant \sum_{n=1}^{\infty} \int_{E_n} g \circ \sup_{x \in E_n} f \, \mathrm{d}\mu \\ &= \sum_{n=1}^{\infty} \int_{E_n} g(n) \, \mathrm{d}\mu = \sum_{n=1}^{\infty} g(n) \mu(E_n) < \infty.\end{aligned}$$

□

Problem: (IV)

Suppose $f_n : \mathbb{R} \to \mathbb{R}$ are nonnegative measurable functions. Prove or disprove by example

$$\limsup_{n\to\infty} \int f_n \mathrm{d}m \leqslant \int \limsup_{n\to\infty} f_n \, \mathrm{d}m.$$

Solution. The claim is false; consider $f_n := n\chi_{[0,1/n]}$ where the integral of each f_n is 1 but $\int \limsup f_n = 0$. □

Problem: (V)

(a) Let $(X, \mathfrak{M}, \mu)$ be a measure space and f an integrable function. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that
$$\mu(A) < \delta \implies \int_A |f| \, \mathrm{d}\mu < \epsilon.$$

(b) For Lebesgue measure m, suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable and $a \in \mathbb{R}$. Define $F(x) := \int_a^x f \, \mathrm{d}m$. Show that F is continuous.

Proof. (1) Since $|f|$ is nonnegative, there exists a sequence of simple functions $\varphi_n \uparrow f$. By the MCT
$$\int_X |f| \, \mathrm{d}\mu = \int_X \lim_{n\to\infty} \varphi_n \, \mathrm{d}\mu = \lim_{n\to\infty} \int_X \varphi_n \, \mathrm{d}\mu.$$

Since f is integrable, there exists a sufficiently large n such that $\int_X |f| - \varphi_n \, \mathrm{d}\mu < \epsilon/2$. If we write φ_n using its standard representation $\sum_{i=1}^k c_i \chi_{E_i}$ and setting $\delta := \epsilon/(2\max c_i)$, then for any A with $\mu(A) < \delta$,
$$\begin{aligned}
\int_X |f| \, \mathrm{d}\mu &= \int_X |f| - \varphi_n \, \mathrm{d}\mu + \int_X \varphi_n \, \mathrm{d}\mu \\
&\leqslant \int_A |f| - \varphi_n \, \mathrm{d}\mu + \int_A \varphi_n \, \mathrm{d}\mu \\
&< \frac{\epsilon}{2} + \sum_{i=1}^k c_i \mu(A \cap E_i) \\
&\leqslant \frac{\epsilon}{2} + \mu(A) \cdot \max c_i < \epsilon.
\end{aligned}$$

(2) Let $\epsilon > 0$ be given. By (1), there exists $\delta > 0$ such that if $m((c,d)) < \delta$ then $\int_c^d |f| \, \mathrm{d}m < \epsilon$. Hence if $x' \in (x - \delta, x)$ we have
$$F(x) - F(x') = \int_{x'}^x f \, \mathrm{d}m \leqslant \int_{x'}^x |f| \, \mathrm{d}m < \epsilon,$$
and similarly if $x'' \in (x, x + \delta)$ we have
$$F(x'') - F(x) = \int_x^{x''} f \, \mathrm{d}m \leqslant \int_x^{x''} |f| \, \mathrm{d}m < \epsilon.$$

This shows precisely that F is continuous. □

Problem: (VI)

Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$ and suppose $F_1, \ldots, F_7$ are sets with $\mu(F_j) \geqslant 1/2$ for all j.

(a) Show that there exist indices $i_1 < i_2 < i_3 < i_4$ for which $F_{i_1} \cap F_{i_2} \cap F_{i_3} \cap F_{i_4} \neq \varnothing$.

(b) Would (a) be correct if we started with 6 measurable sets instead of 7?

Proof. (a) Consider the indicator functions χ_{F_i}. By assumption,
$$\int_X \sum_{i=1}^n \chi_{F_i} \, \mathrm{d}\mu = \sum_{i=1}^7 \int_{F_i} \, \mathrm{d}\mu = \sum_{i=1}^7 \mu(F_i) \geqslant 3.5 > 3.$$

If any four F_i's have empty intersection, then $\sum_{i=1}^{7} \chi_{F_i}$ is bounded above by 3, so

$$\int_X \sum_{i=1}^{n} \chi_{F_i} \, \mathrm{d}\mu \leqslant \int_X 3 \, \mathrm{d}\mu = 3, \text{ contradiction.}$$

(b) No. For example consider $((0,1), \mathcal{B}_{(0,1)}, \mu)$ with $F_1 = F_2 = F_3 = (0, 0.5)$ and $F_4 = F_5 = F_6 = (0.5, 1)$. □

Problem: (VII)

Let $f : [0,1] \to \mathbb{R}$ be continuous. Show that the graph $\{(x, f(x)) : x \in [0,1]\}$ has two-dimensional Lebesgue measure 0.

Proof. Let $\epsilon > 0$ be given. Since f is continuous on a closed interval, it is in particular uniformly continuous, so there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. If we take n sufficiently large so that $1/n < \delta$ and partition $[0,1]$ into n intervals evenly, we see

$$\text{Graph}(f) \subset \bigcup_{i=1}^{n} \left[\frac{i-1}{n}, \frac{i}{n}\right] \times [m_i, M_i]$$

where $m_i = \min_{x \in [(i-1)/n, i/n]} f(x)$ and $M_i = \max_{x \in [(i-1)/n, i/n]} f(x)$. It follows that

$$m(\text{Graph}(f)) \leqslant \sum_{i=1}^{n} \frac{1}{n} \cdot \epsilon = \epsilon.$$

Since ϵ is arbitrary, we are done. □