MATH 525a Homework 5

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Problem: Folland 2.20

If $f_n, g_n, f, g \in L^1$, $f_n \to f$ and $g_n \to g$ a.e., $|f_n| \leqslant g_n$, and $\int g_n \to \int g$, then $\int f_n \to \int f$.

Proof. WLOG assume all functions are real-valued; otherwise we can treat the real and imaginary parts separately. Replacing $g \pm f_n$ by $g_n \pm f_n$, we have $\liminf_{n\to\infty} \int (g_n + f_n) \geqslant \int (g + f)$ and $\liminf_{n\to\infty} \int (g_n - f_n) \geqslant \int (g - f)$. Hence

$$\int g + \liminf \int f_n = \lim \int g_n + \liminf \int f_n = \liminf \left(\int g_n + \int f_n \right) = \liminf \int (g_n + f_n) \geqslant \int (g + f)$$

and

$$\int g - \limsup \int f_n = \lim \int g_n - \limsup \int f_n = \liminf \left(\int g_n - \int f_n \right) = \liminf \int (g_n - f_n) \leqslant \int (g - f).$$

Since $g \in L^1$ we may subtract $\int g$ from both sides and obtain $\liminf \int f_n \geqslant \int f \geqslant \limsup f_n$. That is,

$$\lim_{n\to\infty} \int f_n = \int f.$$

□

Problem: Folland 2.21

Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$.

Proof. For $\Rightarrow$, if $\int |f_n - f| \to 0$ then

$$\left| \int |f_n| - \int |f| \right| \leqslant \int ||f_n| - |f|| \leqslant \int |f_n - f| \to 0.$$

Conversely, define $g_n := |f_n| + |f|, g := 2|f|, h_n := |f_n| - f$, and $h \equiv 0$. Then the assumptions of the previous exercise are met, so $\int h_n \to \int h$, i.e., $\int |f_n - f| \to 0$. □

Problem: (Folland 2.22)

Let μ be the counting measure on $\mathbb{N}$. Interpret Fatou's lemma and the monotone and dominated convergence theorems as statements about infinite series.

Solution. (1) Fatou: if $f_n(k) \geqslant 0$ for all $n, k \in \mathbb{N}$, then

$$\sum_{k\in\mathbb{N}} \liminf_{n\to\infty} f_n(k) \leqslant \liminf_{n\to\infty} \sum_{k\in\mathbb{N}} f_n(k).$$

(2) MCT: if $0 \leqslant f_n(k)$ and $f_n(k) \uparrow f(k)$ for all k then

$$\lim_{n\to\infty} \sum_{k\in\mathbb{N}} f_n(k) = \sum_{k\in\mathbb{N}} f(k).$$

(3) DCT: if $f_n(k) \to f(k)$ for all k, there exists a nonnegative sequence $\{g(k)\}_{k\in\mathbb{N}}$ with $\sum_{k\in\mathbb{N}} g(k) < \infty$, and $|f_n(k)| \leqslant g(k)$ for all n, k, then $\sum_{k\in\mathbb{N}} f(k)$ converges, with

$$\sum_{k\in\mathbb{N}} f(k) = \lim_{n\to\infty} \sum_{k\in\mathbb{N}} f_n(k).$$

Problem: (I)

The right derivative of a function F at t_0 is defined to be

$$F^R(t_0) = \lim_{t\downarrow t_0} \frac{F(t) - F(t_0)}{t - t_0}$$

whenever the limit exists (possibly $\pm\infty$).

Let μ be a measure on $(0, \infty)$ with $\int_{(0,\infty)} \frac{1}{x}\, \mu(\mathrm{d}x) < \infty$ and let $F(t) = \int_{(0,\infty)} \frac{1}{x+t}\, \mu(\mathrm{d}x)$ for $t \geqslant 0$. Show that

$$F^R(0) = -\int_{(0,\infty)} \frac{1}{x^2}\, \mu(\mathrm{d}x)$$

even if this value is $-\infty$.

Proof. Let $t_n \downarrow 0$. The quotient is given by

$$\frac{F(t_n) - F(0)}{t_n - 0} = \int_{(0,\infty)} \frac{1}{t_n}\left(\frac{1}{x+t_n} - \frac{1}{x}\right) \mu(\mathrm{d}x) = -\int_{(0,\infty)} \frac{1}{x(x+t_n)}\, \mu(\mathrm{d}x).$$

Since t_n is positive, $1/(x(x+t_n)) < 1/x^2$. Let $f_n(x) := 1/(x(x+t_n))$ and let $f(x) := 1/x^2$. Then they are all positive with $f_n \uparrow f$. By MCT

$$F^R(0) = \lim_{t_n\downarrow 0} \frac{F(t_n) - F(0)}{t_n} = -\lim_{n\to\infty} \int_{(0,\infty)} f_n\, \mathrm{d}\mu = -\int_{(0,\infty)} f\, \mathrm{d}\mu = -\int_{(0,\infty)} \frac{1}{x^2}\, \mu(\mathrm{d}x).$$

□

Problem: (II)

Let $n \geqslant 1$. Show that

$$g(u) := \int_{-\infty}^{\infty} \frac{x^n e^{ux}}{e^x + 1}\, \mathrm{d}x \qquad u \in (0, 1)$$

is differentiable in $(0, 1)$.

Proof. Let $f(x,u)$ be the integrand; then

$$\frac{\partial f}{\partial u}(x,u) = \frac{x^{n+1}e^{ux}}{e^x+1}.$$

For $u \in (0,1)$ and pick any $a \in (0,u)$ and $b \in (u,1)$. Since

$$\frac{e^{ux}}{e^x-1} \leqslant \begin{cases} e^{-ux} & x<0 \\ e^{(u-1)x} & x \geqslant 0 \end{cases}$$

we have

$$\frac{e^{ux}}{e^x-1} \leqslant \begin{cases} e^{-ax} & x<0 \\ e^{-(1-b)x} & x \geqslant 0. \end{cases}$$

This implies

$$\left|\frac{\partial f}{\partial u}(x,u)\right| \leqslant \begin{cases} |x|^{n+1}e^{-a|x|} & x<0 \\ |x|^{n+1}e^{-(1-b)|x|} & x \geqslant 0. \end{cases}$$

Since both bounds are integrable, $\left|\frac{\partial f}{\partial u}(x,u)\right|$ is bounded by an integrable function. By a theorem in Folland's chapter 2, this shows g is differentiable. □

Problem: (III)

(a) Let $c>0$ and m the Lebesgue measure. Define $\nu(E) := cm(E/c)$. Show that $v = m$.

(b) Let $f \in L^1(\mathbb{R})$ and $c>0$. Show that $\int f(cx)\, m(\mathrm{d}x) = \frac{1}{c}\int f(x)\, m(\mathrm{d}x)$.

(c) Let $f \in L^1(\mathbb{R})$ and $\gamma > 0$ and let $f_n(x) := f(nx)/n^\gamma$ for $n \geqslant 1$. Show that $f_n \to 0$ a.e.

Proof. (a) Since m and ν are Lebesgue-Stieltjes measures, it suffices to check that they agree on the generators, for example the h-intervals. Indeed,

$$\nu((a,b]) = cm((a,b]/c) = cm((a/c,b/c]) = c(b/c-a/c) = b-a = m((a,b]).$$

(b) For indicator $f = \chi_E$:

$$\int \chi_E(cx)\, m(\mathrm{d}x) = \int \chi_{E/c}(x)\, m(\mathrm{d}x) = m(E/c) = \frac{\nu(E)}{c} = \frac{m(E)}{c} = \frac{1}{c}\int \chi_E(x)\, m(\mathrm{d}x).$$

Since integrals are linear, the claim also holds for simple functions. For $f \in L^+$, let $\varphi_n \uparrow f$, each with

$$\int \varphi_n(cx)\, m(\mathrm{d}x) = \frac{1}{c}\int \varphi_n(x)\, m(\mathrm{d}x).$$

Applying MCT to both sides we obtain the equation involving f. Finally, for general $f \in L^1$, we decompose it into f^+ and f^- and the claim would follow.

(c) Define $g(x) := \sum_{n=1}^{\infty} f_n(x)$. To show that $f_n \to 0$ a.e., it suffices to show that $\sum_{n=1}^{\infty} f_n(x) < \infty$ a.e., i.e., $g<\infty$

a.e. It suffices to show $\int g < \infty$. Indeed,

$$\int g(x)\, m(\mathrm{d}x) = \sum_{n=1}^{\infty} \frac{1}{n^\gamma} \int f(nx)\, m(\mathrm{d}x) \leqslant \sum_{n=1}^{\infty} \frac{1}{n^\gamma} \int |f(nx)|\, m(\mathrm{d}x)$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^{1+\gamma}} \int |f(x)| m(\mathrm{d}x) < \infty.$$ □

Problem: (IV)

Suppose $(X, \mathfrak{M}, \mu)$ is a measurable space with $\mu(X) < \infty$ and $f \in L^1(\mu)$ strictly positive. Let $0 < \alpha < \mu(X)$.

(a) Show that
$$\inf\left\{\int_E f\, \mathrm{d}\mu : \mu(E) \geqslant \alpha\right\} > 0.$$

(b) Show by example that (a) can be false if we remove the assumption $\mu(X) < \infty$.

Proof. (a) Since f is strictly positive, $\bigcap_{n=1}^{\infty}\{x : f(x) \leqslant 1/n\} = \varnothing$. This means there exists a sufficiently large N such that
$$\mu(\{x : f(x) < 1/N\}) < \frac{\alpha}{2}.$$
Then if $\mu(E) \geqslant \alpha$,
$$\mu(E \cap \{x : f(x) \geqslant 1/N\}) > \alpha - \frac{\alpha}{2} = \frac{\alpha}{2},$$
so for such E
$$\int_E f\, \mathrm{d}\mu \geqslant \frac{1}{N}\mu(E \cap \{x : f(x) \geqslant 1/N\}) > \frac{\alpha}{2N} > 0.$$
Hence the infimum is strictly positive.

(b) Consider $f(x) = 1/x^2$ on $(1, \infty)$, clearly a L^1 function. For any $\alpha > 0$,
$$\lim_{n\to\infty} \int_{[n,n+\alpha]} f(x)\, \mathrm{d}m = \frac{\alpha}{n(n+\alpha)} \to 0.$$ □

Problem: (V)

Find an example of a sequence $f_n \to 0$ pointwise for which $\int f_n \to 0$ but there is no domination.

Solution. Let $f_n := n^{-1}\chi_{[n-1,n]}$. Then the integral of f_n is $1/n$ which converges to 0, and $f_n \to 0$ pointwise on $[0, \infty)$. However, the supremum of these functions is not L^1, for
$$\int \sum_{n=1}^{\infty} n^{-1}\chi_{[n-1.n]} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Problem: (VI)

Find

$$\lim_{n\to\infty}\int_0^n \left(1+\frac{x}{n}\right)^{-n}\log\left(2+\cos\frac{x}{n}\right)\,\mathrm{d}x.$$

Proof. Since $\cos(x/n) \leqslant 1$ we have $\log(2+\cos x/n) \leqslant \log(3)$. On the other hand, for $0 < x \leqslant n$, using the identity $\log(1+t) \geqslant t - t^2/2$ for $t \in [0,1]$,

$$\left(1+\frac{x}{n}\right)^{-n} = \exp\left[-n\log\left(1+\frac{x}{n}\right)\right] \leqslant \exp\left[-n\left(\frac{x}{n}-\frac{x^2}{2n^2}\right)\right] = \exp\left[-x+\frac{x^2}{2n}\right] \leqslant e^{-x/2}.$$

Since $e^{-x/2}\cdot\log(3)$ is integrable, DCT implies that

$$\begin{aligned}\lim_{n\to\infty}\int_0^n \left(1+\frac{x}{n}\right)^{-n}\log\left(2+\cos\frac{x}{n}\right)\,\mathrm{d}x &= \lim_{n\to\infty}\int_0^\infty \left(1+\frac{x}{n}\right)^{-n}\log\left(2+\cos\frac{x}{n}\right)\chi_{(0,n]}\,\mathrm{d}x\\ &= \int_0^\infty \lim_{n\to\infty}\left[\left(1+\frac{x}{n}\right)^{-n}\log\left(2+\cos\frac{x}{n}\right)\chi_{(0,n]}\right]\mathrm{d}x\\ &= \log(3)\int_0^\infty e^{-x}\,\mathrm{d}x = \log(3).\end{aligned}$$

□

Problem: (VII)

A family $\{f_n\}$ of measurable functions on $(X,\mathfrak{M},\mu)$ is called uniformly integrable if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mu(E) < \delta \implies \int_E |f_n|\,\mathrm{d}\mu < \epsilon \text{ for all } n.$$

Suppose μ is finite, $\{f_n\}$ uniformly integrable, and $f_n \to f$ a.e. Show that $\int |f_n - f|\,\mathrm{d}\mu \to 0$.

Proof. Define $E_n(\epsilon) := \{x : |f_m(x) - f(x)| \leqslant \epsilon \text{ for all } m \geqslant n\}$. Then by a.e. convergence $E_{n-1}(\epsilon) \subset E_n(\epsilon)$ and $\lim_{n\to\infty}\mu(E_n(\epsilon)) = \mu(X)$, so there exists a sufficiently large N such that $\mu(E_N(\epsilon)^c) < \delta$. By uniform integrability,

$$\int_{E_N(\epsilon)^c} |f_n - f|\,\mathrm{d}\mu \leqslant \int_{E_N(\epsilon)^c} |f_n| + |f|\,\mathrm{d}\mu < 2\epsilon,$$

and we also have

$$\int_{E_N(\epsilon)} |f_n - f|\,\mathrm{d}\mu \leqslant \epsilon\mu(E_N(\epsilon)) \leqslant \epsilon\mu(X).$$

Hence for sufficiently large N,

$$\int_X |f_n - f|\,\mathrm{d}\mu = \int_{E_N(\epsilon)^c} |f_n - f|\,\mathrm{d}\mu + \int_{E_N(\epsilon)} |f_n - f|\,\mathrm{d}\mu < 2\epsilon + \epsilon\mu(X) = \epsilon(\mu(X)+2).$$

Since ϵ is arbitrary, we are done. □