MATH 525a Homework 6

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Problem 1: (Folland 2.33)

If $f_n \geqslant 0$ and $f_n \to f$ in measure, show $\int f \leqslant \liminf \int f_n$.

Proof. By Fatou's lemma we know $\int \liminf f_n \leqslant \liminf \int f_n$. First pick a subsequence $\{f_{n_k}\}$ such that

$$\int f_{n_k} \to \liminf \int f_n.$$

Immediately we see $\{f_{n_k}\}$ also converges to f in measure. Using Theorem 2.30, there exists a further subsequence $\{f_{n_{k_j}}\}$ that converges a.e. to f. For this sub-subsequence we also have

$$\int f_{n_{k_j}} \to \liminf \int f_n.$$

Using Fatou's lemma on this sub-subsequence, we have

$$\int f = \int \liminf f_{n_{k_j}} \leqslant \liminf \int f_{n_{k_j}} = \liminf \int f_n. \qquad \square$$

Problem 2: (Folland 2.35)

Show that $f_n \to f$ in measure if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mu(\{x : |f_n(x) - f(x)| \geqslant \epsilon\}) < \epsilon$ for all $n \geqslant N$.

Proof. The $\Rightarrow$ direction follows from the definition of convergence measure: if $f_n \to f$ in measure then $\mu(\{x : |f_n(x) - f(x)| \geqslant \epsilon\}) \to 0$ for all ϵ, so there exists large N after which the corresponding sets all have measure $< \epsilon$. Conversely, the assumption implies a weaker variation:

For every $\epsilon, \delta > 0$ there exists N such that $\mu(\{x : |f_n(x) - f(x)| \geqslant \epsilon\}) < \delta$ for all $n \geqslant N$.

(Simply take $\epsilon' := \min\{\epsilon, \delta\}$ and apply the assumption to ϵ'.) This is precisely the $\epsilon - \delta$ definition showing that $\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geqslant \epsilon\}) = 0$, i.e., $f_n \to f$ in measure. $\square$

Problem 3: (Folland 2.38)

Suppose $f_n \to f$ in measure and $g_n \to g$ in measure.

(a) Show that $f_n + g_n \to f + g$ in measure.

(b) Show that $f_n g_n \to fg$ in measure if $\mu(X) < \infty$ but not necessarily when $\mu(X) = \infty$.

Proof. (a) Using the characterization from 2.35, given $\epsilon > 0$, there exist N_1, N_2 such that

$$\mu(\{x : |f_n(x) - f(x)| \geqslant \epsilon/2\}) < \epsilon/2 \qquad \text{for all } n \geqslant N_1$$

and

$$\mu(\{x : |g_n(x) - g(x)| \geqslant \epsilon/2\}) < \epsilon/2 \qquad \text{for all } n \geqslant N_2.$$

Since $|f_n(x) + g_n(x) - (f+g)| \leqslant |f_n(x) - f(x)| + |g_n(x) - g(x)|$, if the LHS $\geqslant \epsilon$, at least one on the RHS $\geqslant \epsilon/2$. Therefore,

$$\{x : |f_n(x) + g_n(x) - (f(x) + g(x))| \geqslant \epsilon\} \subset \{x : |f_n(x) - f(x)| \geqslant \epsilon/2\} \cup \{x : |g_n(x) - g(x)| \geqslant \epsilon/2\}.$$

Hence, for all $n \geqslant \max\{N_1, N_2\}$, we have

$$\mu(\{x : |f_n(x) + g_n(x) - (f(x) + g(x))| \geqslant \epsilon\}) < \epsilon.$$

This shows $f_n + g_n \to f + g$ in measure.

(b) We first show that if $f_n \to f$ in measure and $\mu(X) < \infty$ then $f_n^2 \to f^2$ in measure.

Proof. Suppose not, that is, there exists $\delta > 0$ and a sequence $\{f_{n_k}\}$ such that $\mu(\{x : |f_{n_k}^2(x) - f(x)^2| \geqslant \epsilon\}) \geqslant \delta$ for all k. Since $\{f_{n_k}\}$ converges to f in measure as well, it has a further subsequence $\{f_{n_{k_j}}\}$ converging to f a.e., so $f_{n_{k_j}}^2 \to f^2$ a.e. as well. Since $\mu(X) < \infty$, Egoroff's theorem states that $f_{n_{k_j}}^2 \to f^2$ almost uniformly. This implies $f_{n_{k_j}}^2 \to f^2$ in measure. (For $\epsilon > 0$ we can pick E with $\mu(E) < \epsilon$ such that the convergence is uniform on E^c, so for large index, the "violation set" is merely E.) However we have assumed that each $f_{n_{k_j}}^2$ has violation $\geqslant \delta$, contradiction. Therefore $f_n^2 \to f^2$ in measure, as claimed. END OF PROOF OF SUBCLAIM

By (a) and the subclaim, $(f_n + g_n)^2 \to (f+g)^2$ in measure, i.e., $f^2 + 2f_n g_n + g_n^2 \to f_n^2 + 2fg + g^2$ in measure. Using (a) again to subtract the squared terms, we have $2f_n g_n \to 2fg$ in measure, so $f_n g_n \to fg$ in measure.

When $\mu(X) = \infty$, this claim easily breaks down. For example, consider $f(x) = g(x) := x$ and $f_n(x) = g_n(x) := x + 1/n$, all defined on $\mathbb{R}$ with the Lebesgue measure. Then the assumptions are met, but

$$|f_n(x)g_n(x) - f(x)g(x)| = \frac{2x}{n} + \frac{1}{n^2} > \frac{2x}{n}.$$

For $\epsilon > 0$ and any n, the set $\{x : |f_n(x)g_n(x) - f(x)g(x)| \geqslant \epsilon\}$ is unbounded from above and therefore has infinite measure, so as $n \to \infty$, the measure does *not* converge to 0, showing that $f_n g_n \not\to fg$ in measure. $\square$

Alternatively, we could do the standard way by noticing that

$$\begin{aligned}\{x : |f_n g_n - fg| > \epsilon\} &\subset \{x : |f_n||g_n - g| > \epsilon/2\} \cup \{x : |g||f_n - f| > \epsilon/2\} \\ &\subset \{|f_n| > M\} \cup \{|g_n - g| > \epsilon/2M\} \cup \{|g| > M\} \cup \{|f_n - f| > \epsilon/2M\}.\end{aligned}$$

By picking a suitable M according to problem (I) below, we can make $\mu(RHS)$ arbitrarily small for large n.

Problem 4: (I)

(a) Suppose $\mu(X) < \infty$ and f is a real-valued measurable function. Then given $\epsilon > 0$ there exists M such that $\mu(\{x : |f(x)| > M\}) < \epsilon$.

(b) Suppose $\mu(X) < \infty$ and f_n, f are real-valued with $f_n \to f$ in measure. Then part (a) can be done uniformly in n: given $\epsilon > 0$, there exists M such that $\mu(\{x : |f_n(x)| > M\}) < \epsilon$ for all n.

Proof. (a) Consider the sets $E_n := \{x : |f(x)| > n\}$. Clearly they are nested, i.e., $E_1 \supset E_2 \supset \ldots$ and $\lim_{n\to\infty} E_n = \varnothing$. Using continuity from above with $\mu(X) < \infty$, we see $\mu(E_n) \to 0$. Therefore, there exists some large M such that $\mu(E_M) = \mu(\{x : |f(x)| > M\}) < \epsilon$.

(b) Note that $|f_n(x)| \leqslant |f(x)| + |f_n(x) - f(x)|$. Hence for all M,

$$\{x : |f_n(x)| > M\} \subset \{x : |f(x)| > M - 1\} \cup \{x : |f_n(x) - f(x)| > 1\}.$$

By convergence in measure, there exists N such that $\mu(\{x : |f_n(x) - f(x)| > 1\}) < \epsilon/2$ for all $n \geqslant N + 1$. There also exists a sufficiently large M for which $\mu(\{x : |f(x)| > M - 1\}) < \epsilon/2$. For $f_1, \ldots, f_n$, by (a) there exists M_i such that $\mu(\{x : |f_i(x)| > M_i\}) < \epsilon$ for $1 \leqslant i \leqslant n$ (i.e., we are bounding the set directly rather than using triangle inequality). Setting $M' := \max\{M, M_1, \ldots, M_n\}$ proves the claim. □

Problem 5: (II)

Suppose f_n, f are measurable functions from $(X, \mathfrak{M}, \mu)$ to $\mathbb{C}$ with $f_n \to f$ in measure.

(a) Show that if $\varphi : \mathbb{C} \to \mathbb{C}$ is uniformly continuous, then $\varphi \circ f_n \to \varphi \circ f$ in measure.

(b) Show that if $\varphi : \mathbb{C} \to \mathbb{C}$ is continuous and $\mu(X) < \infty$, then $\varphi \circ f_n \to \varphi \circ f$ in measure.

(c) Give an example showing that if $\mu(X) = \infty$, you cannot remove the word "uniformly" in (a).

Proof. (a) By uniform continuity, given $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta \implies |\varphi(x) - \varphi(y)| < \epsilon$. That is, if $|\varphi(x) - \varphi(y)| \geqslant \epsilon$ then $|x - y| \geqslant \delta$. Therefore,

$$\{x : |\varphi(f_n(x)) - \varphi(f(x))| \geqslant \epsilon\} \subset \{x : |f_n(x) - f(x)| \geqslant \delta\}.$$

Letting $n \to \infty$, since $f_n \to f$ in measure, the measure of the RHS converges to 0, hence so does the LHS, i.e., $\varphi \circ f_n \to \varphi \circ f$ in measure.

(b) Suppose for contradiction that $\varphi \circ f_n \nrightarrow \varphi \circ f$ in measure. This means that for some $\epsilon > 0$, there exist $\delta > 0$ and a subsequence $\{\varphi \circ f_{n_k}\}$ such that

$$\mu(\{x : |\varphi(f_{n_k}(x)) - \varphi(f(x))| \geqslant \epsilon\}) > \delta \qquad \text{for all } n_k.$$

Since $f_{n_k} \to f$ in measure and $\mu(X) < \infty$, there exists a further subsequence $f_{n_{k_j}}$ converging to f a.e. Since φ is continuous, $\varphi \circ f_{n_{k_j}}$ converges to $\varphi \circ f$ a.e. as well. But then

$$\delta \leqslant \lim_{j\to\infty} (\mu(\{x : |f_{n_{k_j}}(x) - f(x)| \geqslant \epsilon\})) = \lim_{j\to\infty} \int_X \chi_{\text{violation}} \, \mathrm{d}\mu = \int_X \lim_{j\to\infty} \chi_{\text{violation}} \, \mathrm{d}\mu = 0$$

where the interchange of limits is justified by DCT since $\mu(X) < \infty$. Contradiction.

(c) Consider $\mathbb{R}$ with the Lebesgue measure. Let $f(x) := x$, $f_n(x) := x+1/n$, and $\varphi(t) := t^2$. Then $\varphi(f(x)) = x^2$ and $\varphi(f_n(x)) = (x+1/n)^2$. We proved in 2.38(b) that $(x+1/n)^2$ does not converge in measure to x^2. □

Problem 6: (III)

Let f be Lebesgue measurable on $[a, b]$.

(a) Show that there exists a sequence $\{f_n\}$ of continuous functions with $f_n \to f$ almost uniformly on $[a, b]$.

(b) (Lusin's Theorem) Given $\epsilon > 0$, show that there is a set E with $\mu(E^c) < \epsilon$ such that $f\big|_E$ is continuous. In fact one can take E to be compact.

Proof. (a) *This proof needs to assume that $f : [a, b] \to \mathbb{R}$ as opposed to $f : [a, b] \to \overline{\mathbb{R}}$.* Following the hint, define the truncation $f_k := f$ restricted to the set $E_k := \{x : |f(x)| \leqslant k\}$ and let $F_k := E_k^c$. In lecture we showed that continuous functions are dense in L^1, so for each f_k there exists a sequence $\{g_{n,k}\}_{n\geqslant 1}$ of continuous functions converging to f_k in L^1. Then $g_{n,k} \to f_k$ in measure and therefore some subsequence $g_{n_j,k} \to f_k$ a.e. Since $m([a, b]) < \infty$, by Egoroff $g_{n_j,k} \to f_k$ almost uniformly. Now we relabel the functions. For each k, given $\epsilon, \delta > 0$ we can find a continuous function g and a set $S \subset [a, b]$ with $m(S) < \epsilon$ such that

$$\sup\{|f_k(x) - g(x)| : x \in [a, b] - S\} = \sup\{|f(x) - g(x)| : x \in [a, b] - S \cup F_k\} < \delta. \tag{1}$$

Since $\mu(F_k) \to 0$ by continuity from above, there exists a subsequence $\{F_{n_k}\}$ of sets such that $m(F_{n_k}) < 2^{-n-1}$. To each F_{n_k}, almost uniform convergence also guarantees a corresponding S_{n_k} with $m(S_{n_k}) < 2^{-n-1}$. Since δ is also arbitrary, (1) gives the existence of a continuous g_k such that

$$\sup\{|f(x) - g_k(x)| : x \in [a, b] - \underbrace{(E_{n_k} \cup F_{n_k})}_{m(\cdot)<1/2^n}\} < \frac{1}{k}.$$

Let the ϵ corresponding to almost uniform convergence be given. We simply need to pick N sufficiently large so that $m(\bigcup_{k\geqslant N} (E_{n_k} \cup F_{n_k})) \leqslant 2^{-N+1} < \epsilon$. Then for all $k \geqslant N$, $\|f - g_k\|_{\sup} < 1/k \to 0$ on the complement set. This shows $g_k \to f$ almost uniformly.

(b) Let f_n be a sequence of continuous converging almost uniformly to f on $[a, b]$ by according to (a). For $\epsilon > 0$, there exists a set $E \subset [a, b]$ with $m([a, b] - E) < \epsilon/2$ [set-theoretic minus] such that $f_n \to f$ uniformly on E. By a result from MATH 425b, the uniform limit of a sequence of continuous functions is continuous, so $f\big|_E$ is continuous. Also, since m is regular, there exists a compact $K \subset E$ such that $m(E - K) < \epsilon/2$. Then $m([a, b] - K) < \epsilon/2 + \epsilon/2 = \epsilon$ and $f\big|_K$ is continuous, as claimed.

□

Problem 7: (IV)

Let m be Lebesgue measure on $\mathbb{R}$ and let $f_n, f \in L^1(m)$. Suppose there is a constant C such that $\|f_n - f\|_1 \leqslant C/n^2$ for all $n \geqslant 1$. Show that $f_n \to f$ a.e.

Proof. Define $E_n := \{x : |f_n(x) - f(x)| > \epsilon\}$. The bound C/n^2 implies

$$C/n^2 \geqslant \|f_n - f\|_1 = \int_{\mathbb{R}} |f_n - f| \,\mathrm{d}m > \int_{E_n} \epsilon \,\mathrm{d}m = \epsilon\mu(E_n) \implies \mu(E_n) < \frac{C}{\epsilon}\frac{1}{n^2}.$$

Therefore,

$$\mu(E) := \mu(\bigcup_{n=1}^{\infty} E_n) < \frac{C}{\epsilon}\frac{\pi^2}{6}.$$

Now define $F_n := \{x : |f_m(x) - f(x)| > \epsilon \text{ for some } m \geqslant n\}$. Then $F_n = \bigcup_{k\geqslant n} E_k$. Since $\mu(\bigcup_{n=1}^{\infty} E_n) < \infty$, we know $\mu(\bigcup_{k=n}^{\infty} E_k) \to 0$, so $\mu(F_n) \to 0$. Clearly $F_1 \supset F_2 \supset \dots$, and $\mu(F_1) = \mu(\bigcup_{n=1}^{\infty} E_n) = \mu(E) < \infty$. Therefore, continuity from above implies

$$\mu(\bigcap_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \mu(F_n) = 0,$$

i.e.,

$$\mu(\{x : |f_n(x) - f(x)| > \epsilon \text{ infinitely many times}\}) = 0.$$

Since ϵ is arbitrary, the above becomes $\mu(\{x : f_n(x) \nrightarrow f(x)\}) = 0$, i.e., $f_n \to f$ a.e. □

Problem 8: (V)

(a) Suppose $g_1 \geqslant g_2 \geqslant \dots \geqslant 0$ are measurable functions and $g_n \to 0$ in measure. Show that $g_n \to 0$ a.e.

(b) Let $\{f_n\}$ be measurable functions and let $h_n(x) := \sup_{m\geqslant n} |f_m(x) - f_n(x)|$. Show that if $h_n \to 0$ in measure then $\{f_n\}$ converges a.e.

Proof. (a) Since $g_n \to 0$ in measure, there exists a subsequence $\{g_{n_k}\}$ converging to 0 a.e. Let E be the set on which $g_{n_k} \to 0$ so that E^c is a null set.

We now show that $g_n \to 0$ on E. Let $\epsilon > 0$ be given. By convergence of g_{n_k} there exists a sufficiently large n_k such that $|g_{n_k}(x) - 0| = g_{n_k}(x) < \epsilon$. Since $\{g_n\}$ is decreasing, the same inequality holds for all $n \geqslant n_k$. This shows that $g_n \to 0$ on E. Hence $g_n \to 0$ a.e.

(b) Since $h_n \to 0$ in measure, for $\epsilon > 0$ we pick a sequence $\{n_k\}_{k\geqslant 1}$ such that

$$\mu(\{x : h_{n_k}(x) > \epsilon\}) < \epsilon/2^k.$$

Define $\varphi_n := \sup_{k\geqslant n} h_{n_k}$. As n increases, the supremum is taken over a smaller set so φ_n is decreasing. Also, by construction

$$\mu(\{x : \varphi_n(x) > \epsilon\}) < \sum_{k=n}^{\infty} 2^{-k} = 2^{-n+1} \to 0,$$

so $\varphi_n \to 0$ in measure. Then by (a) $\varphi_n \to 0$ a.e., so $\limsup_{m,n\to\infty} |f_m(x) - f_n(x)| = 0$ a.e. Hence f_n converges a.e.[1] □

[1] Credits to Jake for giving me hint on this φ_n.

Problem 9: (VI)

Prove or disprove that for every open $G \subset [0,1]$ the indicator function χ_G is Riemann integrable.

Solution. We disprove the claim using the contrapositive of the following fact mentioned in lecture:

> If f is Riemann integrable, then the Lebesgue integral = the upper and lower Riemann integrals.

Let $\{q_i\}_{i=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [0,1]$ and define G as

$$G := \bigcup_{i=1}^{\infty} (q_i - 2^{-i-2}, q_i + 2^{-i-2})$$

so that G contains all rationals and

$$0 < \mu(G) \leqslant \sum_{i=1}^{\infty} \mu(q_i - 2^{-i-2}, q_i + 2^{-i-2}) = \sum_{i=1}^{\infty} 2^{-i-1} = \frac{1}{2} < 1.$$

Therefore the Lebesgue integral of χ_G is strictly less than 1. However, the upper Riemann integral is 1, as rationals are dense and any subinterval from any partition pair will contain rationals.