MATH 525a Homework 7

Qilin Ye

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Problem 1

Show that for m Lebesgue measure and $f \geqslant 0$ measurable,

$$\int_{\mathbb{R}} f \, \mathrm{d}m = \int_0^\infty m(\{x : f(x) \geqslant t\}) \, \mathrm{d}t.$$

Proof. We can write $m(\{x : f(x) \geqslant t\})$ as

$$\int_{\mathbb{R}} \chi_{\{x:f(x)\geqslant t\}} \, \mathrm{d}m.$$

Then by Tonelli

$$\begin{aligned}\int_0^\infty m(\{x : f(x) \geqslant t\}) \, \mathrm{d}t &= \int_0^\infty \int_{\mathbb{R}} \chi_{\{x:f(x)\geqslant t\}} \, \mathrm{d}m \, \mathrm{d}t \\ &= \int_{\mathbb{R}} \int_0^\infty \chi_{\{x:f(x)\geqslant t\}} \, \mathrm{d}t \, \mathrm{d}m \\ &= \int_{\mathbb{R}} \int_0^{f(x)} \mathrm{d}t \, \mathrm{d}m = \int_{\mathbb{R}} f \, \mathrm{d}m.\end{aligned}$$

□

Problem 2

Exercise 2 on p. 152-153 of the lecutre notes gave an example where $\iint \chi_E \, \mathrm{d}m\mathrm{d}\nu \neq \iint \chi_E \, \mathrm{d}\nu\mathrm{d}m$. In this example, what is $\int_{[0,1]^2} \chi_E \, \mathrm{d}(m \times \nu)$? Equivalently, what is $(m \times \nu)(E)$?

Solution. If E were to be covered by abstract rectangles, it can be equivalently thought as being covered by a countable union of geometric rectangles. However, each geometric rectangle $F = [a, b] \times [c, d]$ has infinite measure (with respect to $m \times \nu$) since

$$(m \times \nu)([a, b] \times [c, d]) = m([a, b]) \cdot \nu([c, d]) = (b - a) \cdot \infty = \infty$$

and taking infimum still implies $(m \times \nu)(E) = \infty$.

Problem 3

Let μ be a finite measure on $\mathbb{R}$ with distribution function F. Show that for $c > 0$,

$$\int_{-\infty}^\infty [F(x + c) - F(x)] \, \mathrm{d}x = c\mu(\mathbb{R}).$$

Proof. For $x > 0$,

$$F(x+c) - F(x) = \mu((0, x+c]) - \mu((0, x]) = \mu((x, x+c])$$

For $-c \leqslant x \leqslant 0$,

$$F(x+c) - F(x) = \mu((0, x+c]) + \mu((x, 0]) = \mu((x, x+c]).$$

Finally, for $x < -c$,

$$F(x+c) - F(x) = -\mu((x+c, 0]) + \mu((x, 0]) = \mu((x, x+c]).$$

In all cases,

$$F(x+c) - F(x) = \int_x^{x+c} \mu(\{x\}) \mathrm{d}\mu.$$

Since $\mu(\cdot)$ is nonengative, by Tonelli,

$$\int_{-\infty}^{\infty} [F(x+c) - F(x)]\, \mathrm{d}x = \int_{-\infty}^{\infty} \int_x^{x+c} \mu(\{x\})\, \mathrm{d}\mu\, \mathrm{d}x = \int_x^{x+c} \int_{-\infty}^{\infty} \mu(\{x\})\, \mathrm{d}x\, \mathrm{d}\mu = c\mu(\mathbb{R}).$$ □

Problem 4

A signed measure λ can be represented as a difference of two positive measures by $\lambda = \lambda^+ - \lambda^-$. Show that λ^+, λ^- are the smallest such measures: if $\lambda = \mu - \nu$ is another representation as a difference of two positive measures (necessarily with at least one of μ, ν finite), then $\mu \geqslant \lambda^+$ and $\nu \geqslant \lambda^-$.

Proof. Let $X = P \cup N$ be the Hahn decomposition of X so that

$$\lambda^+(E) = \lambda(E \cap P) \qquad \lambda^-(E) = \lambda(E \cap N).$$

Firstly, for $E \subset P$, we have

$$\lambda^+(E) = \lambda(E) = \mu(E) - \nu(E) \leqslant \mu(E),$$

so for general $E \subset X$,

$$\lambda^+(E) = \lambda^+(E \cap P) \leqslant \mu(E \cap P) \leqslant \mu(E),$$

from which we see that $\mu \geqslant \lambda^+$. The proof showing $\nu \geqslant \lambda^-$ is entirely analogous and therefore omitted. □

Problem 5

Let μ, ν be signed measures.

(a) Show that a set F is ν-null if and only if the total variation $|\nu|(F) = 0$.

(b) Show that $\nu \perp \mu$ if and only if $|\mu| \perp \mu$.

Proof. (a) First suppose that F is ν-null; that is, if $E \subset F$ then $\mu(E) = 0$. Let $X = P \cup N$ be the Hahn decomposition. By assumption,

$$|\nu|(F) = \nu^+(F) + \nu^-(F) = \nu(F \cap P) + \nu(F \cap N) = 0 + 0 = 0.$$

Conversely, if $|\nu|(F) = 0$, then $\nu^+(F) + \nu^-(F) = \nu(F \cap P) + \nu(F \cap N) = 0$. In particular both are $= 0$ since ν is positive (nonnegative) on P and negative (nonpositive) on N. If $E \subset F$, then $\nu^+(E) \leqslant \nu^+(F)$ and

$\nu^-(E) \leqslant \nu^+(F)$, so $\nu^+(E) = \nu^-(E) = 0$, from which we see $\nu^+(E) - \nu^-(E) = \nu(E) = 0$.

(b) Let $X = P \cup N$ be the Hahn decomposition.

If $\nu \perp \mu$, then there exists a ν-null F whose complement F^c is ν-null. For $E \subset F$,

$$|\nu|(E) = \nu^+(E) - \nu^-(E) = \underbrace{\nu(E \cap P)}_{=0} - \underbrace{\nu(E \cap N)}_{0} = 0.$$

This shows F is also ν-null, so $|\nu| \perp \mu$.

Conversely, if $|\nu| \perp \mu$, then there exists $|\nu|$-null F whose complement F^c is μ-null. Let $E \subset F$. Then

$$|\nu|(F) = \nu^+(F) + \nu^-(F) = \nu(F \cap P) + \nu(F \cap N) = 0.$$

Since ν^+, ν^- are positive (nonnegative) measures, $\nu(F \cap P) = \nu(F \cap N) = 0$. Since $(F \cap P) \sqcup (F \cap N) = F$,

$$\nu(F) = \nu(F \cap P) + \nu(F \cap N) = 0.$$

This proves $\nu \perp \mu$.

□

Problem 6

Let f_n, f be functions on $[0,1]^2$ with measure $m \times m$. Suppose that for almost every fixed x, $f_n(x, \cdot) \to f(x, \cdot)$ a.e. Show that for almost every fixed y, $f_n(\cdot, y) \to f(\cdot, y)$ a.e.

Proof. Let $g_n := f - f_n$ so that $g_n(x, \cdot) \to 0$ a.e. for almost every fixed x. Then, by Tonelli,

$$\int_{[0,1]} \int_{[0,1]} g_n(x,y) \, \mathrm{d}m(y) \, \mathrm{d}m(x) = \int_{[0,1]} \int_{[0,1]} g_n(x,y) \, \mathrm{d}m(x) \, \mathrm{d}m(y) = 0,$$

so $\int_{[0,1]} g_n(x,y) \, \mathrm{d}m(x) = 0$ for almost every y. That is, for such a fixed y, $g_n(x, \cdot) = 0$ for almost every x, i.e., $f_n(x, \cdot) \to f(x, \cdot)$ a.e. □

Problem 7

Let μ be a finite Borel measure on $\mathbb{R}$ and let

$$f(x) := \int_{\mathbb{R}} \frac{\mathrm{d}\mu(y)}{|x-y|^{1/2}}.$$

Here $|x-y|^{-1/2}$ should be interpreted as ∞ when $x = y$. Prove that f is finite a.e. with respect to Lebesgue measure on $\mathbb{R}$.

Proof. It suffices to show that $\int_{\mathbb{R}} f(x) \, \mathrm{d}x < \infty$ (since then f is finite m-a.e.), so it suffices to show that $\int_{[-M,M]} f(x) \, \mathrm{d}x$ is finite for all $M > 0$. Since $1/|x-y|^{1/2}$ is nonnegative and measurable, by Tonelli we have

$$\int_{[-M,M]} f \, \mathrm{d}m = \int_{\mathbb{R}} \int_{[-M,M]} \frac{1}{|x-y|^{1/2}} \, \mathrm{d}m(x) \, \mathrm{d}\mu(y).$$

For $y \in \mathbb{R}$, define $g(y) := \int_{[-M,M]} \frac{1}{|x-y|^{1/2}} \, \mathrm{d}m(x)$. Clearly, if $y > M$ or $y < M$, then they are both bounded above by some constant C_1, say $(2M)^{1/2}$ for example. On the other hand, if $y \in [-M, M]$, then

$$\begin{aligned} g(y) &= \int_{-M}^{y} \frac{1}{|x-y|^{1/2}} \, \mathrm{d}x + \int_{y}^{M} \frac{1}{|x-y|^{1/2}} \, \mathrm{d}x \\ &= \int_{-M}^{y} \frac{1}{(y-x)^{1/2}} \, \mathrm{d}x + \int_{y}^{M} \frac{1}{(x-y)^{1/2}} \, \mathrm{d}x \\ &= 2\sqrt{y+M} + 2\sqrt{M-y} \leqslant 4\sqrt{2M}, \end{aligned}$$

also bounded. Therefore g is bounded on $\mathbb{R}$, and since μ is finite,

$$\int_{[-M,M]} f \, \mathrm{d}m = \int_{\mathbb{R}} g \, \mathrm{d}\mu \leqslant \max\{\sqrt{2M}, 4\sqrt{2M}\}\mu(\mathbb{R}) = 4\sqrt{2M}\mu(\mathbb{R}) < \infty.$$

This proves that f is finite m-a.e. □

Problem 8

Let $(X, \mathfrak{M}, \mu)$ be a measure space and let f_n, f be integrable functions on X. Define measures ν_n, ν on $(X, \mathfrak{M})$ by

$$v_n(A) = \int_A f_n \, \mathrm{d}\mu \qquad \nu(A) = \int_A f \, \mathrm{d}\mu.$$

Suppose that $\sup_{B \in \mathfrak{M}} |\nu_n(B) - \nu(B)| \to 0$. Show that $f_n \to f$ in $L^1(\mu)$.

Proof. Let $\epsilon > 0$. There exists a sufficiently large n_0 such that

$$\sup_{B \in \mathfrak{M}} |\nu_n(B) - \nu(B)| < \epsilon \qquad \text{for all } n \geqslant n_0.$$

Fix $n \geqslant n_0$ and define

$$A := \{x : f(x) > f_n(x)\} \qquad \text{and} \qquad B := \{x : f(x) \leqslant f_n(x)\}.$$

Pick $m \geqslant n_0$ as well. Then we have $A \cap B = \varnothing$, $A \cup B = X$, and

$$\begin{aligned} \int |f - f_n| \, \mathrm{d}\mu &= \left| \int_A (f - f_n) \, \mathrm{d}\mu + \int_B (f_n - f) \, \mathrm{d}\mu \right| \\ &\leqslant \left| \int_A f - f_n \, \mathrm{d}\mu \right| + \left| \int_B f_n - f \, \mathrm{d}\mu \right| < 2\epsilon. \end{aligned}$$

Since ϵ is arbitrary, we are done. □

Problem 9

For a signed measure μ, define $\int f \, \mathrm{d}\nu = \int f \, \mathrm{d}\nu^+ - \int f \, \mathrm{d}\nu^-$ whenever the integrals on the right both exist and are not both infinite. Show that

$$\left| \int f \, \mathrm{d}\nu \right| \leqslant \int |f| \, \mathrm{d}|\nu|.$$

Proof. Let $X = P \cup N$ be the Hahn decomposition of ν. Then

$$\int f \, d\nu = \int f^+ \, d\nu - \int f^- \, d\nu = \int f^+ \, d\nu^+ + \int f^+ \, d\nu^- + \int f^- \, d\nu^+ + \int f^- \, d\nu^-.$$

By assumption, at most one among the four integrals on the RHS is infinite, so

$$\left| \int f \, d\nu \right| \leqslant \int f^+ \, d\nu^+ + \int f^+ \, d\nu^- + \int f^- \, d\nu^+ + \int f^- \, d\nu^- = \int |f^+| \, d\nu^+ + \int |f^-| \, d\nu^- = \int |f| \, d|\nu|$$

□

Problem 10

For a signed measure ν, show that

$$|\nu|(A) = \sup \left\{ \left| \int_A f \, d\nu \right| : |f| \leqslant 1 \right\}.$$

Proof. Let $X = P \cup N$ be the Hahn decomposition of ν. Using the previous problem, for any A and any $|f| \leqslant 1$,

$$\left| \int_A f \, d\nu \right| = \left| \int f\chi_A \, d\nu \right| \leqslant \int |f\chi_A| \, d|\nu| = \int_A \chi_A \, d|\nu|$$

so the $\geqslant$ direction follows. To see that $=$ can be attained, let $f = \chi_{A\cap P} - \chi_{A\cap N}$. Then $|f| = 1$ and

$$\begin{aligned} \int_A \chi_{A\cap P} - \chi_{A\cap N} \, d\nu &= \int_{A\cap P} d\nu - \int_{A\cap N} d\nu = \nu(A\cap P) - \nu(A\cap N) \\ &= \nu^+(A\cap P) + \nu^-(A\cap N) \\ &= \nu^+(A\cap P) + \underbrace{\nu^+(A\cap N)}_{=0} + \nu^-(A\cap N) + \underbrace{\nu^-(A\cap P)}_{=0} = |\nu|(A). \end{aligned}$$

□