MATH 525a Homework 8a

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Problem: (I)

We say two signed measures ν_1, ν_2 on $(X, \mathfrak{M})$ are **compatible** if there exists a decomposition $X = P \cup N$ which is a Hahn decomposition for both measures. According to Proposition 3.14, for ν_1, ν_2 finite signed measures,

$$|\nu_1 + \nu_2|(E) \leqslant |\nu_1|(E) + |\nu_2|(E). \tag{*}$$

Let $\mu = |\nu_1| + |\nu_2|$ and $f_j = \mathrm{d}\nu_j/\mathrm{d}\mu$ for $i = 1, 2$. Show that the following are equivalent:

(1) ν_1 and ν_2 are compatible;

(2) equality holds in (*) for all E; and

(3) $\mu(\{f_1 > 0\} \cap \{f_2 < 0\}) = \mu(\{f_1 < 0\} \cap \{f_2 > 0\}) = 0$.

Proof. ((1) ⇒ (2)). If ν_1, ν_2 are compatible, we let $X = P \cup N$ be the Hahn decomposition for both. Then $\nu_1 + \nu_2$ is positive on P and negative on N, so

$$\begin{aligned}
|\nu_1 + \nu_2|(E) &= (\nu_1 + \nu_2)(E \cap P) - (\nu_1 + \nu_2)(E \cap N) \\
&= \nu_1(E \cap P) - \nu_1(E \cap N) + \nu_2(E \cap P) - \nu_2(E \cap N) \\
&= |\nu_1|(E) + |\nu_2|(E).
\end{aligned}$$

((2) ⇒ (3)). Suppose (3) does not hold; WLOG assume

$$\mu(\{f_1 > 0\} \cap \{f_2 < 0\}) > 0.$$

Then,

$$\begin{aligned}
|\nu_1 + \nu_2|(\{f_1 > 0\} \cap \{f_2 < 0\}) &= \int_{f_1 > 0, f_2 < 0} |f_1 + f_2| \, \mathrm{d}\mu \\
&< \int_{f_1 > 0, f_2 < 0} |f_1| + |f_2| \, \mathrm{d}\mu \\
&= \int_{f_1 > 0, f_2 < 0} |f_1| \, \mathrm{d}\mu + \int_{f_1 > 0, f_2 < 0} |f_2| \, \mathrm{d}\mu \\
&= |\nu_1|(\{f_1 > 0\} \cap \{f_2 < 0\}) + |\nu_2|(\{f_1 > 0\} \cap \{f_2 < 0\}).
\end{aligned}$$

Thus the contrapositive has been proven. Finally, to show ((3) ⇒ (1)), define

$$P := \{f_1 \geqslant 0\} \cap \{f_2 \geqslant 0\} \qquad \text{and } N := P^c.$$

It is clear that ν_1, ν_2 are both positive on P. To show that ν_1, ν_2 are both negative on N (and thus $P \cup N$ is a Hahn decomposition), it suffices to show that ν_1 is negative on $\{f_1 < 0\} \cap \{f_2 \geqslant 0\}$ and $\{f_1 \geqslant 0\} \cap \{f_2 < 0\}$ (the argument for ν_2 is analogous). Indeed, on one hand

$$\nu_1(E) = \int_E f_1 \, \mathrm{d}\mu < 0 \qquad \text{for all } E \subset \{f_1 < 0\} \cap \{f_2 \geqslant 0\}.$$

On the other hand, note that

$$S := \{f_1 \geqslant 0\} \cap \{f_2 < 0\} = (\{f_1 > 0\} \cap \{f_2 < 0\}) \cup (\{f_1 = 0\} \cap \{f_2 < 0\}) =: S_1 \cup S_2.$$

For any $E \subset S$, define $E_1 := E \cap S_1$ and $E_2 := E \cap S_2$. Then

$$\begin{aligned} \nu_1(E) &= \int_E f_1 \, \mathrm{d}\mu = \int_{E_1} f_1 \, \mathrm{d}\mu + \int_{E_2} f_1 \, \mathrm{d}\mu \\ &= \int_{\mu\text{-null set}} f_1 \, \mathrm{d}u + \int_{E_2} 0 \, \mathrm{d}\mu = 0. \end{aligned}$$

Therefore ν_1 is negative on N. This completes the proof, as $P \cup N$ is indeed a Hahn decomposition. □

Problem: (VI)

(1) Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be σ-finite. Suppose the measure λ on $\mathfrak{M} \times \mathfrak{N}$ satisfies $\lambda(A \times B) = \mu(A)\nu(B)$ for all $A \in \mathfrak{M}, B \in \mathfrak{N}$. Show that $\lambda = \mu \times \nu$ on $\mathfrak{M} \times \mathfrak{N}$.

(2) Suppose μ_1, ν_1 are σ-finite on $(X, \mathfrak{M})$ and μ_2, ν_2 are σ-finite on $(Y, \mathfrak{N})$. Suppose $\nu_1 \ll \mu_1$ and $\nu_2 \ll \mu_2$. Show that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{\mathrm{d}(\nu_1 \times \nu_2)}{\mathrm{d}(\mu_1 \times \mu_2)}(x, y) = \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}(x) \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}(y).$$

Proof. (1) Let $\mathcal{A}$ be the collection of finite union of abstract rectangles in $\mathfrak{M} \otimes \mathfrak{N}$. From the fact that $\lambda(A \times B) = \mu(A)\nu(B)$ we see that λ defines a premeasure on $\mathcal{A}$. Since $\mathfrak{M} \otimes \mathfrak{N}$ is the σ-algebra generated by $\mathcal{A}$, by the Carathéodory's theorem, λ can be *uniquely* extended to a measure on $\mathfrak{M} \otimes \mathfrak{N}$. Since $\mu \times \nu$ is one such measure, we must have $\lambda = \mu \times \nu$.

(2) Let f, g be the corresponding Radon-Nikodym derivatives $\mathrm{d}\nu_1/\mathrm{d}\mu_1$, $\mathrm{d}\nu_2/\mathrm{d}\mu_2$, respectively. If $A \times B \subset \mathfrak{M} \times \mathfrak{N}$, then

$$\begin{aligned} (\nu_1 \times \nu_2)(A \times B) &= \nu_1(A_1)\nu_2(A_2) \\ &= \int_A f_1 \, \mathrm{d}\mu_1 \int_B f_2 \, \mathrm{d}\mu_2 = \int \chi_A f_1 \, \mathrm{d}\mu_1 \int \chi_B f_2 \, \mathrm{d}\mu_2 \\ &= \iint f_1 f_2 \chi_{A \times B} \, \mathrm{d}\mu_1 \mathrm{d}\mu_2 = \int f_1 f_2 \chi_{A \times B} \, \mathrm{d}(\mu_1 \times \mu_2) \end{aligned}$$

by definition and Tonelli. Thus for abstract rectangles of form $A \times B$, we have $\nu_1 \times \nu_2 = (f_1 f_2)(\mu_1 \times \mu_2)$. By linearity the same holds for *finite* unions of abstract rectangles, which form an algebra $\mathcal{A}$ that generates $\mathfrak{M} \otimes \mathfrak{N}$. By the first part, this extends to all of $\mathfrak{M} \times \mathfrak{N}$. This $(f_1 f_2)$ implies $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{\mathrm{d}(\nu_1 \times \nu_2)}{\mathrm{d}(\mu_1 \times \mu_2)}(x, y) = f_1(x) f_2(y) = \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}(x) \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}(y).$$ □

Problem: (VII)

Suppose $\nu \ll \mu$ are finite positive measures and let $\rho = \mu + \nu$, $f = \mathrm{d}\mu/\mathrm{d}\rho$, $g = \mathrm{d}\nu/\mathrm{d}\rho$.

(1) Show that $f + g = 1$ and $f > 0$ μ-a.e.

(2) Show that $\mathrm{d}\nu/\mathrm{d}\mu = g/f$.

Proof. For all E, we have

$$\rho(E) = \mu(E) + \nu(E) = \int_E f \,\mathrm{d}\rho + \int_E g \,\mathrm{d}\rho = \int_E f + g \,\mathrm{d}\rho = \int_E 1 \,\mathrm{d}\rho.$$

Since E is arbitrary we see $f + g = 1$ ρ-a.e., that is, the exception takes place on a set F with $\rho(F) = 0$. However, $\rho(F) = \mu(F) + \nu(F)$, so this implies $\nu(F) = 0$, i.e., $f + g = 1$ μ-a.e.
From above we see that $\mu \ll \rho$. Also, if $\mu(E) = 0$, since $\nu \ll \mu$ we have $\nu(E) = 0$ so $\rho(E) = 0 + 0 = 0$. This implies $\rho \ll \mu$ as well. Since both measures are finite, by chain rule

$$\frac{\mathrm{d}\rho}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\rho} = \frac{\mathrm{d}\rho}{\mathrm{d}\rho} = 1 \qquad \mu\text{-a.e. and } \rho\text{-a.e.}$$

f clearly cannot be negative on a set of positive ρ-measure (otherwise the integral evaluates to a negative number while also corresponding to the μ-measure of a set), and the claim holds verbatim replacing ρ-measure by μ-measure. Along with the equation above, we see that $f > 0$ a.e.

For (2), using the identity above and the fact that $\nu \ll \lambda$ as well (clearly as $\ll$ is transitive),

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \frac{\mathrm{d}\nu}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}\mu} = \frac{g}{f}.$$ □

Problem: (VIII)

For a Borel set $E \subset \mathbb{R}^n$, its density $\rho_E(x)$ at x is defined to be

$$\rho_E(x) := \lim_{r\to 0} \frac{m(E \cap B(r,x))}{m(B(r,x))}$$

whenever it exists.

(1) If E is a square and x a corner, what is $\rho_E(x)$?

(2) Show that for a.e. x, the limit exists and has value 0 or 1. Up to a null set, what is the set where the limit is 1?

(3) Suppose $E_1 \subset E_2 \subset \ldots$ and $E := \bigcup_{n\geqslant 1} E_n$. Is it necessarily true that $\rho_{E_n} \to \rho_E$ a.e.? If $\rho_{E_n}(x)$ and $\rho_E(x)$ exist for all x, is it necessarily true that $\rho_{E_n} \to \rho_E$ pointwise?

Proof. (1) $1/4$.

(2) By Theorem 3.18, since $\chi_E \in L^1_{\mathrm{loc}}$,

$$\lim_{r\to 0} A_r\chi_E(x) = \lim_{r\to 0} \frac{1}{m(B(r,x))}\int_{B(r,x)} \chi_E(y)\,\mathrm{d}y = \chi_E(x)$$

for a.e. $x \in \mathbb{R}^n$. Rewriting the definition above, we have

$$\lim_{r\to 0} \frac{m(E \cap B(r,x))}{m(B(r,x))} = \chi_E(x).$$

Hence the limit = 1 for a.e. $x \in E$ and = 0 for a.e. $x \in E^c$.

(3) For the first case, yes. By assumption $\chi_{E_n} \to \chi_E$ as $n \to \infty$. By the previous part, $\rho_E(x) = 1$ a.e. for $x \in E$ and $\rho_E(x) = 0$ a.e. for $x \notin E$. Hence $\rho_E \equiv \chi_E$ a.e. and similarly $\rho_{E_n} \equiv \chi_{E_n}$ a.e. Therefore $\chi_{E_n} \to \chi_E$ implies $\rho_{E_n} \to \rho_E$ a.e.

For the second case, no. Consider $E_1 := \mathbb{R} - [-1,1]$ and similarly $E_n := \mathbb{R} - [-1/n, 1/n]$. Then $E_1 \subset E_2 \subset \dots$ and $\bigcup_{n\geqslant 1} E_n = \mathbb{R} - \{0\}$. Also set $x = 0$. For each E_n, the numerator eventually becomes 0 for sufficiently small r, so the limit $\rho_{E_n}(0) = 0$. However, for E, the ratio is always 1 since

$$m(E \cap B(r,0)) = m(B(r,0) - \{0\}) = m(B(r,0)).$$

Hence $\rho_E(0) = 1$, providing a counterexample.

□