MATH 525a Homework 8b

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Problem 2

Let $f : [0,1] \to \mathbb{R}$ satisfy

$$|f(x) - f(y)| \leqslant |\sqrt{x} - \sqrt{y}|$$

for all $x, y \in [0,1]$. Is f necessarily absolutely continuous?

Solution. Yes. For $x, y \in [0,1]$ we have $|x - y| \leqslant |x + y|$, so for $\{(x_i, y_i)\}_{i=1}^n$ finitely many disjoint intervals $\subset [0,1]$,

$$\begin{aligned}\sum_{i=1}^n |f(y_i) - f(x_i)| &\leqslant \sum_{i=1}^n |\sqrt{y_i} - \sqrt{x_i}| = \sum_{i=1}^n \sqrt{|\sqrt{y_i} - \sqrt{x_i}|^2} \\ &\leqslant \sum_{i=1}^n \sqrt{|\sqrt{y_i} - \sqrt{x_i}||\sqrt{y_i} + \sqrt{x_i}|} = \sum_{i=1}^n \sqrt{|y_i - x_i|} \leqslant \sum_{i=1}^n |y_i - x_i|.\end{aligned}$$

Therefore, we can simply pick $\delta := \epsilon$ in the (ϵ, δ) definition to show that f is absolutely continuous.

Problem 3

(1) Show that there exists an increasing function F on $\mathbb{R}$ such that $F' = 0$ a.e. but F is not constant on any open interval.

(2) Let E be a closed subset of $\mathbb{R}$ which is null for Lebesgue measure, and let G be a bounded, absolutely continuous, strictly increasing function on $\mathbb{R}$. Show that the function F in (a) can be chosen so $F = G$ on the set E.

(3) Show that the function in (a) can be chosen so $F'(0)$ exists and is not 0.

Proof. (1) We consider a right-continuous F so that μ_F is well-defined. By assumption $\mu_F = 0$ a.e., so μ_F lives on a m-null set. Conversely, since F is not constant on any open interval, μ_F cannot be identically 0 on any interval, so μ_F lives on a dense set of $\mathbb{R}$. To this end, we consider a μ_F that lives on $\mathbb{Q}$. Let $\{q_n\}$ be an enumeration of the rationals and define $\mu_F(x) := 2^{-n}\delta_{q_n}(x)$. Then

$$F(x) := \sum_{q_j \leqslant x} 2^{-j}$$

is the function we seek.

(2) Since E is closed, its complement is open. We claim E^c is dense, for otherwise there exists $x \in \mathbb{R}$ and

$\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \cap E^c = \varnothing$, implying $(x-\epsilon, x+\epsilon) \subset E$ so $m(E) \geqslant 2\epsilon > 0$, contradiction. Therefore E^c is a countable union of intervals. By density, these intervals must be of form $\{(a_i, b_i)\}_{i\geqslant 1}$.

The result from (a) implies the existence of a function $\varphi : [0,1) \to [0,1)$ that is increasing with zero derivative a.e. but non-constant on any open interval (for we can take rationals only in $[0,1]$ and define φ analogously). Likewise we can define such function from $\varphi(x; a, b, c, d) : [a, b) \to [c, d)$ via translation and scaling.

Note that

$$F(x) := \varphi(x; a_i, b_i, G(a_i), G(b_i)) \qquad \text{where } x \in [a_i, b_i)$$

is increasing and satisfies $F(x_j) = G(x_j)$ for all j so $F = G$ on E. Also, by (a) φ has derivative 0 except on a null set on (x_{i-1}, x_i) so F has derivative 0 a.e. on $\mathbb{R}$. It is also clear that F must be non-constant on any open interval as each φ is.

□

Problem 4

Suppose f is absolutely continuous on $[0,1]$. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that $m(A) < \delta$ implies $m^*(f(A)) < \epsilon$. In particular, this means that $m(A) = 0$ implies $m(f(A)) = 0$. We use outer measure m^* because the continuous image of a measurable set need not be measurable.

Proof. Let $\epsilon > 0$ be given and let $\delta > 0$ be the corresponding bound given by the definition of absolute continuity. We claim that the same δ works for the statement given in the question. Let A be such that $m(A) < \delta$. Then there exist disjoint open intervals $\{(a_i, b_i)\}_{i\geqslant 1} \subset [0,1]$ such that

$$A \subset \bigcup_{i=1}^{\infty}(a_i, b_i) \qquad \text{and } m(A) \leqslant m(\bigcup_{i=1}^{\infty}(a_i, b_i)) = \sum_{i=1}^{\infty} m((a_i, b_i)) < \delta.$$

For each (a_i, b_i), the continuous image of closed $[a_i, b_i]$ is compact, so there exist $c_i \leqslant d_i$ such that

$$\{f(c_i), f(d_i)\} = \{\sup_{[a_i,b_i]} f(x), \inf_{[a_i,b_i]} f(x)\}.$$

Therefore,

$$m(f(a_i, b_i)) = \sup_{[a_i,b_i]} f(x) - \inf_{[a_i,b_i]} f(x) = |f(d_i) - f(c_i)|.$$

Then, we have

$$\begin{aligned} m^*(f(A)) &\leqslant m(f(\bigcup_{i=1}^{\infty}(a_i, b_i))) \leqslant \sum_{i=1}^{\infty} m(f(a_i, b_i)) \\ &= \sum_{i=1}^{\infty} |f(d_i) - f(c_i)| < \epsilon \end{aligned}$$

as

$$\bigcup_{i=1}^{\infty}(c_i, d_i) \subset \bigcup_{i=1}^{\infty}(a_i, b_i) \qquad \text{and } m(\bigcup_{i=1}^{\infty}(a_i, b_i)) < \delta.$$

This shows the claim and we are done. □

Problem 5

If f and g are absolutely continuous on $[a,b]$ with $g \neq 0$, show that f/g is absolutely continuous.

Proof. Let $\epsilon > 0$ be given and let $\delta > 0$ be the corresponding bound that satisfies the definition for *both* the absolutely continuity of f and that of g.

Since $[a,b]$ is compact and f, g are continuous, both images are compact and in particular bounded. Define

$$M_f := \sup_{[a,b]} |f(x)| \qquad \text{and} \qquad m_g := \inf_{[a,b]} |g(x)|.$$

Notice that

$$\begin{aligned} \sum_{i=1}^{n} \left| \frac{f(b_i)}{g(b_i)} - \frac{f(a_i)}{g(a_i)} \right| &= \sum_{i=1}^{n} \left| \frac{f(b_i)}{g(b_i)} - \frac{f(b_i)}{g(a_i)} + \frac{f(b_i)}{g(a_i)} - \frac{f(a_i)}{g(a_i)} \right| \\ &\leqslant \sum_{i=1}^{n} |f(b_i)| \left| \frac{1}{g(b_i)} - \frac{1}{g(a_i)} \right| + \sum_{i=1}^{n} \frac{1}{|g(a_i)|} |f(b_i) - f(a_i)| \\ &\leqslant \frac{M_f}{m_g^2} \sum_{i=1}^{n} |g(b_i) - g(a_i)| + \frac{1}{m_g} \sum_{i=1}^{n} |f(b_i) - f(a_i)|. \end{aligned} \tag{Δ}$$

Therefore if a finite disjoint union $\bigcup_{i=1}^{N} (a_i, b_i)$ has measure $< \infty$,

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| \qquad \text{and} \qquad \sum_{i=1}^{n} |g(b_i) - g(a_i)|$$

both $< \epsilon$, so $(\Delta) < \epsilon(M_f/m_g^2 + 1/m_g)$. Since ϵ is arbitrary and $M_f/m_g^2 + 1/m_g$ is just a constant, we are done. ☐