MATH 525a Homework 9a

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Problem: (Folland 3.36)

Let G be a continuous increasing function on $[a,b]$ and let $G(a)=c, G(b)=d$.

(a) If $E\subset[c,d]$ is a Borel set, then $m(E)=\mu_G(G^{-1}(E))$.

(b) if f is a Borel measurable and integrable function on $[c,d]$, then

$$\int_c^d f(y)\,\mathrm{d}y=\int_a^b f(G(x))\,\mathrm{d}G(x).$$

In particular,

$$\int_c^d f(y)\,\mathrm{d}y=\int_a^b f(G(x))G'(x)\,\mathrm{d}x$$

if G is absolutely continuous.

(c) The validity of (b) may fail if G is merely right continuous than continuous.

Proof. (a) First we assume E is an interval of form $[c_0,d_0]$. Since G is continuous on $[a,b]$, by IVT there exist $a_0<b_0$ such that $G(a_0)=c_0$ and $G(b_0)=d_0$. Also, for $c_0\leqslant y\leqslant d_0$, by IVT we have $G(y)=x$ for some $y\in[a,b]$, and by monotonicity of G, we further have $y\in[a_0,b_0]$. Hence $G^{-1}([c_0,d_0])=[a_0,b_0]$. Therefore, for intervals $E=[c_0,d_0]$,

$$m(E)=d_0-c_0=G(b_0)-G(a_0)=\mu_G([a_0,b_0])=\mu_G(G^{-1}(E)). \tag{1}$$

Next, if $m(E)=\mu_G(G^{-1}(E))$, then

$$\begin{aligned} m(E^c)&=m([c,d])-m(E)=d-c-\mu_G(G^{-1}(E))\\ &=\mu_G([a,b])-\mu_G(G^{-1}(E))=\mu_G([c,d]\backslash G^{-1}(E))\\ &=\mu_G((G^{-1}(E))^c).\qquad\text{(with respect to }[c,d]\text{)}\end{aligned}$$

To restate the result,

$$m(E)=\mu_G(G^{-1}(E))\implies m(E^c)=\mu_G((G^{-1}(E))^c). \tag{2}$$

Finally, let us first note that if $E_1\cap E_2=\varnothing$, then $G^{-1}(E_1)=G^{-1}(E_2)=\varnothing$. Now if $\{E_n\}_{n\geqslant1}$ is a countable collection of disjoint subsets of $[c,d]$ with $\mu(E_n)=\mu_G(G^{-1}(E_n))$ for all n, then

$$m(\bigcup_{n\geqslant1}E_n)=\sum_{n=1}^\infty m(E_n)=\sum_{n=1}^\infty\mu_G(G^{-1}(E_n))=\mu_G(G^{-1}(\bigcup_{n\geqslant1}E_n)) \tag{3}$$

since m and μ_G are both measures and thus countably additive. (2), (3), along with the trivial fact that $m([c,d]) = \mu_G(G^{-1}([c,d]))$ and $m(\varnothing) = \mu_G(G^{-1}(\varnothing))$ imply that the collection

$$\{E : m(E) = \mu_G(G^{-1}(E))\}$$

forms a σ-algebra. Thus, the general principle along with (1) implies that all sets in the σ-algebra generated by intervals on $[a,b]$, i.e., $\mathcal{B}_{[a,b]}$ have the same property.

(b) If f is an indicator function χ_E for $E \in \mathcal{B}_{[a,b]}$, then

$$\int_c^d \chi_E(y)\,\mathrm{d}y = m(E) = \mu_G(G^{-1}(E)) = \int_a^b \chi_{G^{-1}(E)}(x)\,\mathrm{d}\mu_G(x) = \int_a^b \chi_E(G(x))\,\mathrm{d}G(x)$$

so the claim holds for indicator functions and thus simple functions (with standard representation). Next, for $f \in L^+([c,d])$, if we let φ_n be a sequence of indicator functions $\nearrow f$, then

$$\int_c^d f(y)\,\mathrm{d}y = \int_c^d \lim_{n\to\infty} \varphi_n\,\mathrm{d}m = \lim_{n\to\infty} \int_c^d \varphi_n\,\mathrm{d}m = \lim_{n\to\infty} \int_a^b \varphi_n(G(x))\,\mathrm{d}G(x) = \int_a^b f(G(x))\,\mathrm{d}G(x).$$

Finally, for a general f, the result follows from decomposing f into f^+ and f^-.

In particular, if G is absolutely continuous then G is differentiable a.e. with $\mathrm{d}G(x) = G'(x)$ a.e. so

$$\int_c^d f(y)\,\mathrm{d}y = \int_a^b f(G(x))\,\mathrm{d}G(x) = \int_a^b f(G(x))G'(x)\,\mathrm{d}x.$$

(c) Let $[a.b] := [-1,1]$, $[c,d] := [0,1]$, and $G := \chi_{[0,1]}$, an increasing, right-continuous function on $[-1,1]$ with $G(-1) = 0$, $G(1) = 1$. Finally, let $f(x) := x$ on $[0,1]$. Then $\int_0^1 f(x)\,\mathrm{d}x = \frac{1}{2}$ whereas

$$\int_{-1}^1 f(G(x))\,\mathrm{d}G(x) = \int_{-1}^0 0\,\mathrm{d}G(x) + \int_0^1 1\,\mathrm{d}G(x) = 1.$$ □

Problem: (I)

It can be shown, and you may assume, that there exists a measurable set $E \subset [0,1]$ with $0 < m(E \cap I) < m(I)$ for all interval $I \subset [0,1]$ with $m(I) > 0$. Let $F(x) = m(E \cap [0,x])$ for $x \in [0,1]$. Show that F is absolutely continuous and strictly increasing, but F^{-1} is not absolutely continuous.

Proof. First we show F is strictly increasing. Suppose not, i.e., for some $0 \leqslant x < y \leqslant 1$ we have

$$F(x) = m(E \cap [0,x]) = m(E \cap [0,y]) = F(y).$$

Taking the difference we have $m(E \cap [x,y]) = 0$ whereas $m([x,y]) > 0$, contradiction.

Next we show F is absolutely continuous. Let $\epsilon > 0$ be given and we let $\delta = \epsilon$. Then if $\{(a_i, b_i)\}$ are disjoint subsets of $[0,1]$ with total length $< \epsilon$, we have

$$\sum_{i=1}^n |F(b_i) - F(a_i)| = \sum_{i=1}^n m(E \cap (a_i, b_i]) < \sum_{i=1}^n m((a_i, b_i)) = \sum_{i=1}^n |b_i - a_i|.$$

Finally, to show F^{-1} is not absolutely continuous, note by homework 8 problem VIII,

$$\lim_{h\searrow 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\searrow 0} \frac{m(E \cap (x, x+h])}{h} \leqslant \lim_{r\to 0} \frac{2m(E \cap B(r,x))}{B(r,x)} = \chi_E(x)$$

and likewise for $\lim_{h\nearrow 0}$, so F is differentiable with $F'(x) = 0$ a.e. on $F := [0,1]\backslash E$. We know however that F has positive measure since $m(E) = m(E \cap [0,1]) < 1$. The proof consists of two steps.

(Step 1). We first show that $m(f(F)) = 0$.
Let $\epsilon > 0$ be given. We define

$$F_n := \{x \in F : |x-y| < 1/n \Rightarrow |f(x) - f(y)| < \epsilon|x-y|\}. \tag{1}$$

For all $x \in F$, since $f'(x) = 0$, the derivative quotient $\dfrac{|f(x)-f(y)|}{x-y} \to 0$ as $|x-y| \to 0$. Thus, each $x \in F$ is contained in some F_n, i.e., $F = \bigcup_{n=1}^{\infty} F_n$. It is also clear that $F_1 \subset F_2 \subset \cdots$, so by continuity from below,

$$m(F) = \lim_{n\to\infty} m(F_n). \tag{2}$$

From $F_1 \subset F_2 \subset \cdots$ we obtain $f(F_1) \subset f(F_2) \subset \cdots$ and from $F = \bigcup_{n=1}^{\infty} F_n$ we obtain $f(F) = \bigcup_{n=1}^{\infty} f(F_n)$. Therefore, using continuity from below once more,

$$m^*(f(F)) = \lim_{n\to\infty} m^*(f(F_n)). \tag{3}$$

For each n, there exists a collection $\{I_k\}$ of intervals with each $m(I_k) < 1/n$ and $\sum_{k=1}^{\infty} m(I_k) < m(F_n) + \epsilon$. Then,

$$m^*(f(I_k)) = |f(\sup I_k) - f(\inf I_k)| < \epsilon m(I_k) \qquad \text{by (1)},$$

so

$$m^*(f(F_n)) \leqslant m^*(\bigcup_{k=1}^{\infty} f(I_k)) \leqslant \sum_{k=1}^{\infty} m^*(f(I_k)) < \epsilon \sum_{k=1}^{\infty} m(I_k) < \epsilon(m(F_n) - \epsilon).$$

Taking $n \to \infty$, we have

$$m^*(f(F)) = \lim_{n\to\infty} m^*(f(F_n)) \leqslant \lim_{n\to\infty} \epsilon(m(F_n) - \epsilon) = \epsilon(m(F) - \epsilon).$$

Since ϵ is arbitrary but $m(F)$ finite in our case, letting $\to 0$ implies $m^*(f(F)) = m(f(F)) = 0$.
(Step 2). By homework 8b problem 4, if f^{-1} is absolutely continuous, then $m(f^{-1}(f(F))) = 0$, but $m(f^{-1}(f(F))) = m(F)$ and we know $m(F) > 0$. Therefore f^{-1} is not absolutely continuous. □

Problem: (II)

Let $F : [a,b] \to \mathbb{R}$ be continuous. Let $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ be a partition of $[a,b]$ and let F_P be the function with $F_P(x_i) = F(x_i)$ and linear in between then x_i's. We define a local minimum to be a pair (x,y) with $x \in (a,b)$ and $y = F(x)$ such that there exists an interval (u,v) containing x with $F \geqslant y$ on (u,v). A local maximum is defined analogously. Let

$$N_F(y) = |\{x : F(x) = y\}| \qquad \text{and } N_{F,mm}(y) = |\{x : (x,y) \text{ is a local max/min}\}|$$

where $|\cdot|$ denotes the cardinality. It is given that both sets are measurable.

(a) Show that there are at most countably many local maxima and minima, so $N_{F,mm} = 0$ m-a.e.

(b) Show that for any partition P,

$$\int_{\mathbb{R}} N_{F_P}(y)m(\mathrm{d}y) = T_{F_P}([a,b]) \leqslant T_F([a,b])$$

where T_{F_P}, T_F denote the total variation of F_P and F on $[a,b]$.

(c) Suppose $P(1) \subset P(2) \subset \dots$ are partitions with mesh $\to 0$. Show that for all y with $N_{F,mm}(y) = 0$ we have $N_{F_{P(n)}}(y) \nearrow N_F(y)$.

(d) Show that

$$\int_{\mathbb{R}} N_F(y)m(\mathrm{d}y) = T_F([a,b]).$$

Proof. (a) Suppose for contradiction that F takes uncountably many extrema *values*. That is,

$$\bigcup_{n\geqslant 1}[\{y = F(x) : F \leqslant y \text{ on } (x-n^{-1}, x+n^{-1})\} \cup \{y = f(x) : F \geqslant y \text{ on } (x-n^{-1}, x+n^{-1})\}]$$

is uncountable. Therefore for some m, the corresponding union is uncountable, and WLOG we assume

$$E_n := \{y = F(x) : F \leqslant y \text{ on } (x - 1/m, x + 1/m)\} \qquad \text{is uncountable.}$$

Since F is continuous on a compact and connected domain, the range of F is connected and bounded. In particular E_n must have a limit point y_0, and suppose $\{y_k\}_{k\geqslant 1} \subset E_n$, a sequence of distinct points in E_n, converges to y_0. Define each x_k to be any number in $[a,b]$ such that $f(x_k) = y_k$. By compactness of $[a,b]$ there exists a further subseqeuence k_j of indices such that

$$y_{k_j} \to y_0, \qquad y_{k_j} = f(x_{k_j}), \qquad x_{k_j} \to \text{some } x_0, \qquad \text{and } f(x_0) = y_0.$$

Let N be sufficiently large so that $|x_{k_{j_1}} - x_{k_{j_2}}| < 1/n$ for $k_{j_1}, k_{j_2} > N$. Then, since $y_{k_{j_1}} \neq y_{k_{j_2}}$, either $y_{k_{j_1}}$ fails to be a local maximum on $(x_{k_{j_1}} - n^{-1}, x_{k_{j_2}} + n^{-1})$ or $y_{k_{j_2}}$ fails on $(x_{k_{j_2}} - n^{-1}, x_{k_{j_2}} + n^{-1})$. Either way, we have a contradiction against the definition of E_n. Therefore F can take at most countably many extrema *values*, i.e., $N_{F,mm} = 0$ m-a.e.

(b) For $1 \leqslant i \leqslant n$, define a function p_i by

$$p_i(y) := \begin{cases} 1 & y \text{ is between } F(x_{i-1}), F(x_i) \text{ (inclusive)} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\sum_{i=1}^{n} p_i(y)$ agrees with $N_{F_P}(y)$, except possible at the endpoints of the parition, where overcounting may occur. However, since there are only finitely many partition points, they do not affect the Lebesgue integral. Thus,

$$\int_{\mathbb{R}} N_{F_P}(y)m(\mathrm{d}y) = \sum_{i=1}^{n} \int_{\mathbb{R}} p_i(y)m(\mathrm{d}y) = \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = T_{F_P}([a,b]).$$

By definition, T_F is taken over the supremum of all partitions, so $T_{F_P}([a,b]) \leqslant T_F([a,b])$.

(c) See the proof of 3(d). *I originally skipped 3(c), but I accidentally proved it when attempting 3(d).*

(d) By MCT,

$$\int_{\mathbb{R}} N_F(y)m(\mathrm{d}y) = \int_{\mathbb{R}} \lim_{n\to\infty} N_{F_{P(n)}}(y)m(\mathrm{d}y) = \lim_{n\to\infty} \int_{\mathbb{R}} N_{F_{P(n)}}(y)m(\mathrm{d}y)$$
$$= \lim_{n\to\infty} T_{F_{P(n)}}([a,b]).$$

It remains to show that

$$\lim_{n\to\infty} T_{F_{P(n)}}([a,b]) = \sup_P T_{F_P}([a,b]) = T_F([a,b]).$$

Let $\epsilon > 0$ and $P' = \{a = x'_0 < x'_1 < \ldots < x'_k = b\}$ be a partition with $T_{F_{P'}}([a,b]) > T_F([a,b]) - \epsilon$. Since F is continuous on $[a,b]$, it is uniformly continuous, so there exists $\delta_0 > 0$ such that

$$|x-y| < \delta_0 \implies |f(x) - f(y)| < \frac{\epsilon}{k+1}.$$

Let $\delta := \min\{\delta_0, \mathrm{mesh}(P')\}$ and let n be sufficiently large so that $\mathrm{mesh}(P(n)) < \delta$. We write $P(n)$ as $\{a = x_0 < x_1 < \ldots < x_n = b\}$. Since $\mathrm{mesh}(P(n)) < \mathrm{mesh}(P')$, each subtinterval of $P(n)$ can contain at most one partition point in P'. Define

$$\mathcal{J} := \{1 \leqslant k \leqslant n : (x_{k-1}, x_k) \text{ contains some } x'_i \in P'\}. \tag{Δ}$$

Then $|\mathcal{J}| \leqslant \min(n, k+1) \leqslant k+1$ and

$$T_{F_{P(n)}} = \sum_{k=1}^{n} |F(x_k) - F(x_{k-1})| = \sum_{k\in\mathcal{J}} |F(x_k) - F(x_{k-1})| + \sum_{k\notin\mathcal{J}} |F(x_k) - F(x_{k-1})|.$$

Now define $P'' := P(n) \cup P'$, a refinement of both partitions. It follows that $T_{F_{P''}}([a,b]) \geqslant T_{F_{P'}}([a,b]) > T_F([a,b]) - \epsilon$. Also, (following the notation in (Δ)),

$$T_{F_{P''}}([a,b]) = \sum_{k\in\mathcal{J}} \Big[|F(x_k) - F(x'_i)| + |F(x'_i) - F(x_{k-1})|\Big] + \sum_{k\notin\mathcal{J}} |F(x_k) - F(x_{k-1})|.$$

Therefore,

$$\begin{aligned} T_{F_{P(n)}} &\geqslant \sum_{k\notin\mathcal{J}} |F(x_k) - F(x_{k-1})| \\ &= T_{F_{P''}}([a,b]) - \sum_{k\in\mathcal{J}} \Big[|F(x_k) - F(x'_i)| + |F(x'_i) - F(x_{k-1})|\Big] \\ &> T_{F_{P''}}([a,b]) - 2\epsilon|\mathcal{J}| > T_F([a,b]) - \epsilon - \frac{2\epsilon(k+1)}{k+1} = T_F([a,b]) - 3\epsilon. \end{aligned}$$

Since ϵ is arbitrary, we are done. □