MATH 525a Homework 9b

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Problem: (Folland 5.4)

If X, Y are normed vector spaces, then the map $(T, x) \mapsto Tx$ is continuous from $\mathcal{L}(X,Y) \times X$ to Y. That is, if $T_n \to T$ and $x_n \to x$ then $T_n x_n \to Tx$.

Proof. Let $T_n \to T$ (i.e., for all x, $T_n x \to Tx$) and let $x_n \to x$. By triangle inequality,

$$\|T_n x_n - Tx\| \leqslant \|T_n x_n - T_n x\| + \|T_n x - Tx\| \leqslant \|T_n\|_{\text{op}} \|x_n - x\| + \|T_n x - Tx\|.$$

The assumption that $T_n \to T$ implies the norms $\|T_n\|$ is a bounded sequence. Since we have $\|x_n - x\| \to 0$ and $\|T_n x - Tx\| \to 0$ as $n \to \infty$, taking limsup we see $\limsup_{n\to\infty} \|T_n x_n - Tx\| = 0$ so $T_n x_n \to Tx$ as claimed. □

Problem: (I)

Consider $C([0,1])$ equipped with $\|\cdot\|_{\sup}$. Let $R \subset C([0,1])$ be the collection of continuously differentiable functions g with $g(0) = 0$. Define $T : C([0,1]) \to C([0,1])$ by $(Tf)(x) := \int_0^x f(t)\, \mathrm{d}t$.

(a) Show that the range of T is R and T is a bijection between $C([0,1])$ and R.

(b) Show that T is bounded, find its norm, and show that T^{-1} is not bounded.

Proof. (a) Clearly every $Tf \in R$ according to the fundamental theorem of calculus, and for any $g \in R$, the continuous derivative g' satisfies $Tg' = g$, so T is surjective. If $Tf_1 = Tf_2$ then $T(f_1 - f_2)$ is the zero function, so

$$\int_0^y f_1(t) - f_2(t)\, \mathrm{d}t = 0 \qquad \text{for all } y \in [0,1].$$

Suppose f_1 is not identical to f_2. WLOG assume for some $x \in [0,1]$ we have $f_1 > f_2$. By continuity there exists an open interval $(x - \epsilon, x + \epsilon)$ on which $f_1 > f_2$. Thus,

$$\int_0^{x+\epsilon} f_1(t) - f_2(t)\, \mathrm{d}t - \int_0^{x-\epsilon} f_1(t) - f_2(t)\, \mathrm{d}t = \int_{x-\epsilon}^{x+\epsilon} f_1(t) - f_2(t)\, \mathrm{d}t > 0,$$

so $\int_0^{x+\epsilon} (f_1 - f_2)$ and $\int_{x-\epsilon}^{x+\epsilon} (f_1 - f_2)$ cannot be both zero, contradiction. Therefore T is injective.

(b) The operator norm of T is 1: on one hand,

$$\begin{aligned}|(Tf)(x)| &= \left|\int_0^x f(t)\,\mathrm{d}t\right| \leqslant \int_0^x |f(t)|\,\mathrm{d}t \\ &\leqslant \int_0^1 |f(t)|\,\mathrm{d}t \leqslant \int_0^1 \|f\|_{\sup}\,\mathrm{d}t = 1\cdot\|f\|_{\sup}\end{aligned}$$

which implies $\|T\|_{\text{op}} \leqslant 1$, whereas on the other hand, if we feed any constant function to T, the norm 1 is attained. Hence $\|T\|_{\text{op}} = 1$.

We now show that T^{-1} is unbounded. Consider $g_n(x) := \sin(nx)$. From (a) we know $g_n = Tf_n$ where $f_n(x) = g_n'(x) = n\cos(nx)$. While all g_n's have sup norm 1, $\|f_n\|_{\sup} = n \to \infty$ as $n \to \infty$. This shows that T^{-1} is unbounded. ☐

Problem: (II)

Show that ℓ^1 equipped with $\|\cdot\|_{\sup}$ is not complete.

Proof. Notice that $\ell^1 \subset \ell^\infty$, i.e., if $\sum_n |x_n| < \infty$ then $\sup|x_n| < \infty$. In ℓ^∞ equipped with $\|\cdot\|_{\sup}$, the sequence $x^{(n)} := (1, 1/2, \ldots, 1/n, 0, 0, \ldots)$ converges to $x := (1, 1/2, 1/3, \ldots)$, but $x \notin \ell^1$ as the harmonic series is not summable. Thus, $\{x_n\}$ is Cauchy but does not have a limit in ℓ^1, i.e., $(\ell^1, \|\cdot\|_{\sup})$ is incomplete. ☐

Problem: (III)

Let V be a subspace of a NLS X and suppose V contains some open ball $B(x,r)$. Show that $V = X$.

Proof. By definition of a subspace, if $y \in B(x,r)$ and $x, y \in V$, then $x + n(y-x) \in V$ for all n. Letting y vary we see $B(x, nr) \subset V$ for all n. But $\bigcup_{n=1}^{\infty} B(x, nr) = X$, so $V \supset X$ and we are done. ☐

Problem: (IV)

Let V be a subspace of a NLS X and let W be Banach. Let $T : V \to W$ be a bounded linear map. Show that there exists a unique extension $\tilde{T}$ of T defined on $\overline{V}$ with $\|\tilde{T}\|_{\text{op}} = \|T\|_{\text{op}}$.

Proof. Since $\tilde{T}$ needs to be continuous, for each $x \in \overline{X} - V$, we define $\tilde{T}x := \lim \tilde{T}x_n = \lim Tx_n$, where $\{x_n\}$ is any sequence converging to x. Also, this definition is well-defined: on one hand Tx_n is Cauchy in W so the limit exists; on the other hand, if $\{x_n\}, \{x_n'\}$ both converge to x, then $\lim_{n\to\infty} \|x_n - x_n'\| = 0$ and so

$$\lim_{n\to\infty} Tx_n - \lim_{n\to\infty} Tx_n' = \lim_{n\to\infty} T(x - x_n') = T(0) = 0,$$

i.e., $\lim Tx_n = \lim Tx_n'$.

To show $\tilde{T}$ is linear, let $x, y \in \overline{V}$ and let $\{x_n\}, \{y_n\}$ be points in V converging to x and y, respectively. Then by continuity of $\tilde{T}$, for $\lambda \in \mathbb{R}$,

$$\tilde{T}(\lambda x + y) = \lambda\tilde{T}x + \tilde{T}y = \lambda \lim_{n\to\infty} Tx_n + \lim_{n\to\infty} Ty_n = \lambda\tilde{T}x + \tilde{T}y.$$

To show uniqueness, if $\tilde{T}$, $\tilde{T}'$ are both extensions, then $\tilde{T}x \neq \tilde{T}'x$ for some $x \in \overline{V} - V$. But this contradicts the uniqueness of $\lim Tx_n$.

Finally, to show that $\|\tilde{T}\|_{\text{op}} = \|T\|_{\text{op}}$, let us note that

$$\|\tilde{T}\|_{\text{op}} = \sup_{\substack{x\in\overline{V} \\ \|x\|=1}} \|\tilde{T}x\| \geqslant \sup_{\substack{x\in V \\ \|x\|=1}} \|Tx\| = \|T\|_{\text{op}}.$$

Suppose $\|\tilde{T}\|_{\text{op}} > \|T\|_{\text{op}}$. By definition of supremum there exists $x \in \overline{V}$ with $\|x\| = 1$ such that

$$\|\tilde{T}\|_{\text{op}} \geqslant \|\tilde{T}x\| \qquad \text{with } \|\tilde{T}\|_{\text{op}} - \|\tilde{T}x\| < (\|\tilde{T}\|_{\text{op}} - \|T\|_{\text{op}})/2.$$

On the other hand, there exists a sequence $\{x_n\}$ converging to x, so in particular $Tx_n = \tilde{T}x_n \to \tilde{T}x$ as shown before. Hence there exists a sufficiently close x_n such that

$$\|\tilde{T}x - \tilde{T}x_n\| = \|Tx - \tilde{T}x_n\| < (\|\tilde{T}\|_{\text{op}} - \|T\|_{\text{op}})/2.$$

But then

$$\begin{aligned} \|\tilde{T}\|_{\text{op}} - \|\tilde{T}x_n\| &= \|\tilde{T}\|_{\text{op}} - \|\tilde{T}x\| + \|\tilde{T}x\| - \|\tilde{T}x_n\| \\ &\leqslant \|\tilde{T}\|_{\text{op}} - \|\tilde{T}x\| + \|\tilde{T}x - \tilde{T}x_n\| < \|\tilde{T}\|_{\text{op}} - \|T\|_{\text{op}}, \end{aligned}$$

implying that $\|Tx_n\| > \|T\|_{\text{op}}$, a contradiction to the definition of supremum. Hence $\|\tilde{T}\|_{\text{op}} = \|T\|_{\text{op}}$. □