Beginning of Aug.25, 2021

Hausdorff ⇒ Zorn's. let X be as in Zorn's lemma (i.e., every lienarly ordered subset has an upper bound). By Hausdorff, X has a maixmal linearly ordered E (which cannot be enlarged). Zorn's assumption says that E has an upper bound. Call it e. We claim that e is maximal.

Suppose for contradiction that e is not maximal, so there existss y such that $y \geqslant e$. But then the set $E \cup \{y\}$ is linearly ordered and bigger than E, contradicting the Hausdorff assumption. Therefore no such y exists, i.e., e is maximal.

For the converse, see p.5 of text. □

Example 0.0.1: Application of Zorn's Lemma. Let V be an infinite dimensional vector space, W a subspace, and $f : W \to \mathbb{R}$ a linear functional. Can we always extend f to all of V (still linear)? Note that if V is finite dimensional this is obvious.

Solution. Yes; first notice that we can always extend by one dimension. Take some $x \notin W$ and choose any value $f(x) = b$. For every $y \in \operatorname{span}\{W \cup \{x\}\}$ has form $y = w + cx$ with $w \in W$. Then we can extend f by

$$f(y) = f(w + cx) := f(w) + cb.$$

It is easy to verify that f has been extended and is still linear.

Now we generalize. A definition first. If W_1, W_2 are subspace of V and $g_i : W_i \to \mathbb{R}$, we say (W_2, g_2) is an **extension** of (W_1, g_1) if

$$W_2 \supset W_1 \qquad \text{and} \qquad g_2 = g_1 \text{ on } W_1.$$

(We write $(W_2, g_2) \geqslant (W_1, g_1)$.) This is a partial order on

$$\mathcal{W} := \{\text{all linear functionals on subspaces of } V\}.$$

Now we check conditions for Zorn's lemma. Does every linearly ordered subset of V has an upper bound?

Suppose $\mathcal{E} = \{(W_\alpha, g_\alpha) : \alpha \in A\}$ is a linearly ordered subset, meaning that any two W_α, W_β, one always contains the other. We consider

$$W_{\text{up}} := \bigcup_{\alpha \in A} W_\alpha \qquad \text{and} \qquad g_{\text{up}} := g_\alpha \text{ on } W_\alpha.$$

Then $(W_{\text{up}}, g_{\text{up}})$ is an upper bound for all for $\mathcal{E}$. Thus the assumption of Zorn's lemma is satisfied.

Therefore by Zorn's lemma $\mathcal{W}$ contains a maximal $(W_{\max}, g_{\max})$. But $W_{\max}$ must be all of W, otherwise we could extend it by one dimension as mentioned above. This proves the claim!

□

Theorem 0.0.2: The Well-Ordering Principle

Every nonempty set X can be well ordered.

Proof. Let $\mathcal{W}$ be the set of all subsets $W \subset E \times E \subset X \times X$ where E some subset of X and W is an ordering on E. (In other words we are considering an ordering of orderings here, and soon we will attempt to find a "maxiamal" ordering that hopefully applies to all of X, thereby making it a well-ordered set.) We can order $\mathcal{W}$ (partially) via $W_1 \leqslant W_2$ if W_1 well-orders some E_i $(i = 1, 2)$ and W_2 extends the ordering "upward":

(1) $E_1 \subset E_2$ and the 2 orders agree on E_1 (meaning $W_1 \subset W_2$ as sets), and

(2) In the ordering W_2, all elements of e_1 are $\leqslant$ all elements in $E_2 - E_1$.

We want to apply Zorn's lemma. Check hypothesis: suppose that some collection $\mathcal{W}^* = \{W_\alpha : \alpha \in A\} \subset \mathcal{W}$ is linearly ordered. Namely, any two orderings in $\mathcal{W}^*$ satisfy (1) and (2). Then the union of all these sets, $\bigcup_\alpha E_\alpha$, is our candidate for upper bound. Pick $x, y \in \bigcup_\alpha E_\alpha$. Is $x \leqslant y$? Pick $x \in E_\alpha$ and $y \in E_\beta$. One of E_α, E_β is bigger, say the latter. Then both $x, y \in E_\beta$ and the ordering of E_β will determine either $x \leqslant y$ or $y \leqslant x$. □