Beginning of Oct.29, 2021

Weak Solutions

Define $C_0^\infty(\Omega)$ to be $\{f \in C^\infty(\Omega) : \operatorname{supp} f \subset \Omega \text{ compact}\}$.

We define two different inner products:

$$\langle u, v\rangle_0 := \int_\Omega u \cdot v \, \mathrm{d}x$$

$$\langle u, v\rangle_1 := \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$$

We can check that $C_0^\infty(\Omega)$ is an inner product space under $\langle \cdot, \cdot\rangle_0$ or under $\langle \cdot, \cdot\rangle_1$. Define $\|u\|_0^2 := \langle u, u\rangle_0$ and $\|u\|_1^2 := \langle u, u\rangle_1$.

Lemma: Poincaré Inequality

For all $u \in C_0^\infty(\Omega)$, $\|u\|_0 \leqslant d\|u\|_1$ where d is the diameter of Ω.

Proof. Since $u\big|_{\partial\Omega} = 0$, at any point $x \in \Omega$, we have by FTC

$$u(x) = \int_{x^b}^{x} \partial_{x_1} u \, \mathrm{d}x_1$$

where $x^b \in \partial\Omega$ and all but the first component of x^b agrees with that of x. Then, by Hölder's inequality,

$$u^2(x) = \left(\int_{x^b}^{x} 1 \cdot \partial_{x_1} u \, \mathrm{d}\tilde{x}_1\right)^2 \leqslant d \cdot \left(\int_{x^b}^{x} |\partial_{x_1} u|^2 \, \mathrm{d}\tilde{x}_1\right).$$

Integrating over Ω gives

$$\int_\Omega u^2 \, \mathrm{d}x \leqslant d \int_\Omega \left(\int_{x^b}^{x_1} |\partial_{x_1} u|^2 \, \mathrm{d}\tilde{x}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n \leqslant d^2 \int_\Omega |\nabla u|^2 \, \mathrm{d}x.$$

□

Definition 0.0.1: $H_0^1(\Omega)$

We define $H_0^1(\Omega)$ to be the completion of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_1$. This is a Hilbert space.

Lemma

Each $u \in H_0^1(\Omega)$ belongs to $L^2(\Omega)$ and has partial derivatives $u_j \in L^2(\Omega)$. Furthermore,

$$\langle v, u_j\rangle_0 = -\langle \partial v / x_j, u\rangle_0 \qquad \text{for all } v \in C_0^\infty(\Omega).$$

Moreover,

$$\langle u, v\rangle_1 = \int_\Omega u_j v_j \, \mathrm{d}x$$