Beginning of Nov.17, 2021

Proof. (1)

(2) We now show that graph of A is closed. To this end, by Lax-Milgram, for $f \in L^2(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ such that

$$B[u,v] = \langle f, v\rangle_{L^2} \qquad \text{for all } u \in H_0^1(\Omega).$$

Then, the map $f \mapsto u = L^{-1}f$ is a bounded linear operator from $L^2(\Omega)$ to $L^2(\Omega)$. Note that $(u,f) \in$ Graph(A) is equivalent to $(-f,u) \in$ Graph(L^{-1}). Since L^{-1} is continuous, its graph is closed.

(3) We want to show that for all $\lambda > 0$, the operator $\lambda I - A$ has a bounded inverse $\|(\lambda I - A)^{-1}\| \leqslant 1/\lambda$. That is, it suffices to show that for each $f \in L^2(\Omega)$, the problem

$$\begin{cases} \lambda u + Lu = f & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

has a unique solution satisfying

$$(\|(\lambda I - A)' f\|_{L^2} =) \qquad \|u\|_{L^2} \leqslant \frac{\|f\|_{L^2}}{\lambda}$$

By Lax-Milgram, there exists a unique $u \in H_0^1(\Omega)$ such that

$$\langle \lambda u, u\rangle_{L^2} + B[u,v] = \langle f, v\rangle_{L^2} \qquad \text{for all } v \in H_0^1(\Omega).$$

Taking $v = u$ gives

$$\lambda\|u\|_{L^2} + B[u,u] = \langle f, u\rangle_{L^2} \leqslant \|f\|_{L^2}\|u\|_{L^2}$$

Therefore $\lambda\|u\|_{L^2} \leqslant \|f\|_{L^2}$, and this proves the claim $\|(\lambda I - A)^{-1}\| \leqslant 1/\lambda$.

Therefore, by the semigrouop theory, we deduce that the linear operator A generates a contractive semigroup. □

General Operators

Example 0.0.1. Let A be an $m \times n$ matrix. Consider the linear ODE on $\mathbb{R}^n$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -Ax(t).$$

If A is PD, then $-A$ generates a contractive semigroup. In particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 = 2\left\langle \frac{\mathrm{d}x(t)}{\mathrm{d}t}, x(t)\right\rangle = 2\langle -Ax(t), x(t)\rangle \leqslant 0.$$

Hence $|x(t)|^2 \leqslant |x(0)|^2$ for all $t \geqslant 0$, giving a contractive bound.

Now let A be any matrix. Then we can let γ be sufficiently large such that $A + \gamma I$ is PD. Then the claim above holds analogously, as $-(A+\gamma I)$ generates a contractive semigroup. If $x(t) = e^{-tA}x(0)$ is a solution, then

$$\begin{aligned} |x(t)| &= |e^{-tA}x(0)| = |e^{(\gamma I - (A+\gamma I))t}x(0)| \\ &= e^{\gamma t}|e^{-(A+\gamma I)x(0)}| \leqslant e^{\gamma t}|x(0)|. \end{aligned}$$

Hence $-A$ generates a semigroup of type γ.

Example 0.0.2. Consider a general elliptic operator where the corresponding bilinear form $B[u,v]$ is not necessarily PD. In this case, one can also find a sufficiently large $\gamma > 0$ such that

$$B_\gamma[u,v] := B[u,v] + \gamma \langle u,v\rangle_{L^2}$$

is strictly PD on $H_0^1(\Omega)$. Define $L_\gamma u := Lu + \gamma I$. Then $u_t = -Lu$ can be written as $u - t = -L_\gamma u + \gamma u$. By the previous theorem, $A_\gamma := -(L+\gamma I)$ generates a contractive semigroup of linear oeprators $\{S_t^{(\gamma)} : t \geqslant 0\}$. Now $A = -L = \gamma I - L_\gamma$ with $D(A) = \{u \in H_0^1(\Omega); Lu \in L^2(\Omega)\}$ generates a semigroup of type γ with

$$\{S_t : t \geqslant 0\} \text{ where } S_t = e^{\gamma t} S_t^{(\gamma)}, t \geqslant 0.$$

What about the solution map $t \mapsto u(t) = S_t g$?

(1) For all initial data $g \in L^2(\Omega)$, $t \mapsto u(t) = S_t g$ is C^1 on $D(A)$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = Lu(t) \qquad \text{for all } t > 0.$$

(2) The map $t \mapsto u(t) = S_t g$ is continuous from $[0,\infty)$ into $L^2(\Omega)$ and satisfies $u(0) = g$. Thus, the initial condition is satisfied as an identity in L^2 (because S_0 is).

(3) If $g \in D(A)$, then $u(t) = S_t g \in D(A)$ for all $t \geqslant 0$. Then the map $t \mapsto u(t)$ is continuously differentiable and

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u = Au \\ u(0) = g \end{cases} \qquad \text{for all } t > 0.$$

Since $D(A) \subset H_0^1(\Omega)$, this implies $u(t)$ satisfies the correct boundary condition for all $t \geqslant 0$.

(4) If $g \in L^2(\Omega)$, then there exists a sequence $g_n \in D(A)$ such that $\|g - g_n\|_{L^2} \to 0$ as $n \to \infty$. If the semigroup is of type γ, we have

$$\|S_t g_n - S_t g\|_{L^2} \leqslant e^{\gamma t} \|g_n - g\|_{L^2} \to 0$$

for fixed g. Hence $u \mapsto u(t) = S_t f$ is the limit of of a sequence of C^1 solutions $t \mapsto u_n t = S_t g_n$.

Claim. $u = u(t,x)$ provides a solution to the parabolic equation

$$u_t = \sum_{i,j=1}^{n} (a^{i,j} u_{x_i})_{x_j} - \sum_{i=1}^{n} b^i u_{x_i} - cu$$

in the sense of distribution: for all $\varphi \in C_c^\infty(\Omega \times (0,\infty))$,

$$\iint_\Omega u\varphi_t + \sum_{i,j=1}^{n} u(a^{i,j}\varphi_{x_j})_{x_i} + \sum_{i=1}^{n} u(b^i\varphi)_{x_i} - cu\varphi \,\mathrm{d}x\mathrm{d}t = 0.$$

Consider $u_m := S_t g_m$ where $\|g - g_m\|_{L^2} \to 0$. $t \mapsto u_m(t)$ converges to the continuous trajectory $t \mapsto u(t) = S_t g$ in $C^0([0,t]; L^2(\Omega))$. We deduce that u is a solution in the sense of distribution.