0.1   Introduction

$$\text{Linear Algebra} \implies \text{Functional Analysis} \Rightarrow \begin{cases} \text{Calculus of Variations} \\ \text{ODEs/PDEs} \\ \text{Approximation Theory} \\ \dots \end{cases}$$

(1)   Functional analysis deals with *infinite* dimensional spaces, often spaces of *functions* and certain classes of *mappings* between these spaces.

(2)   Can be thought of a (nontrivial) generalization of linear algebra (from finite to infinite dimensional).

(3)   The development of functional analytic methods and results are important in various areas of math; in particular they provide a powerful tool for the study of ODEs, PDEs, for example the existence and uniqueness of solutions and the convergence of approximations, etc.

0.2   Linear Algebra

We first draw an analogy between functional analysis and linear algebra, starting from familiar concepts. Let A be a $n \times n$ matrix.

(p1) Given a vector $b \in \mathbb{R}^n$, find $x \in \mathbb{R}^n$ such that $Ax = b$.

(p2) Analogy: *consider a bounded, open set $\Omega \subset \mathbb{R}^n$ and a linear partial differential operator*

$$Lu = -\sum_{i,j=1}^{n} (a^{i,j}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u.$$

Given a function $f : \Omega \to \mathbb{R}$, find a function u vanishing on the boundary of Ω so that $Lu = f$.

Fundamental differences between (p1) and (p2)

The matrix A is a *continuous* (or equivalently *bounded*) linear transformation on the *finite* dimensional space $\mathbb{R}^n$, whereas the differential operator L is considered an *unbounded* (and equivalently *not continuous*) operator on *infinite* dimensional spaces $L^2(\Omega)$. In particular, the domain of L is not $L^2(\Omega)$, but only a suitable subspace (e.g. $H_0^1(\Omega)$, the space of functions vanishing on the boundary of Ω with $\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |u|^2 \, \mathrm{d}x + \int_\Omega |\nabla u|^2 \, \mathrm{d}x \right)^{1/2} < \infty$).

Structural Similarities

(1)   (Positivity) If A is strictly positive definite, i.e., there exists $\beta > 0$ such that $\langle Ax, x \rangle \geqslant \beta \|x\|^2$ for all $x \in \mathbb{R}^n$, then A is invertible as its kernel is trivial. If so, (p1) has a *unique* solution for all $b \in \mathbb{R}^n$.

Analogy in elliptic PDEs: if L is strictly positive definite, i.e.,

$$\langle Lu, u \rangle_{L^2} := \int_\Omega Lu \cdot u \, \mathrm{d}x = \int_\Omega \sum_{i,j=1}^{n} a^{i,j}(x)u_{x_i}u_{x_j} + \sum_{j=1}^{n} b^j(x)u_{x_j}u + c(x)u^2 \, \mathrm{d}x \geqslant \beta \|u\|_{H_1^0(\Omega)}^2$$

for some $\beta > 0$ and for all $u \in H_0^1(\Omega)$, then (p2) has a unique solution $u \in H_0^1(\Omega)$ for every $f \in L^2(\Omega)$. In particular, at each $x \in \Omega$, the $n \times n$ matrix $\{a^{i,j}(x)\}$ should be strictly positive definite.

(2) (Fredholm Alternative) In linear algebra, (p1) has a unique solution if and only if $Ax = 0$ has one solution $x = 0$. In other words, uniqueness comes if and only if the kernel is trivial. In general this is *false*; if X is infinite dimensional, one can construct a bounded linear operator $\Lambda : X \to X$ that is injective but not surjective (or vice versa).

However, there is a remedy: if Λ is of a specific form given by $\Lambda = I - K$ where I is the identity and K a compact operator, then Λ is injective if and only if its surjective. Consequently, (p2) has a unique solution if and only if the homogeneous equation $Lu = 0$ only has a trivial solution.