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0.1 Evolution Equations

Consider an ODE

$$\begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = b \end{cases}$$

where A is a matrix and $b \in \mathbb{R}^n$. Linear ODE theory gives $x(t) = e^{tA}b$ where

$$e^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}.$$

Also, $\{e^{tA} \mid t \in \mathbb{R}\}$ has the group property

$$e^{0A} = I \quad \text{and} \quad e^{tA} e^{sA} = e^{(t+s)A}.$$

If A is symmetric, there exists an orthonormal basis of eigenvectors $\{v_1, \dots, v_n\}$ with eigenvalues $\lambda_1, \dots, \lambda_n$ such that

$$e^{tA}b = \sum_{k=1}^n e^{t\lambda_k} \langle b, v_k \rangle v_k.$$

Now we use linear semigroup theory and extend the above notion to linear operators in infinite dimensional spaces.

Consider

$$\begin{cases} \frac{d}{dt} u(t) = -Lu(t) \\ u(0) = g & \text{for } g \in L^2(\Omega) \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

(For example, consider $u_t = \Delta u$ the heat equation.) When $a^{i,j}(x) = a^{j,i}(x)$ and $b^i(x) = 0$, then the elliptic operator L is symmetric (like how A is a symmetric matrix above), and the solution can be decomposed along orthogonal basis $\{\varphi_1, \varphi_2, \dots\}$ of $L^2(\Omega)$:

$$u(t) = \sum_{k=1}^{\infty} e^{-t\lambda_k} \langle g, \varphi_k \rangle_{L^2} \varphi_k := S_t g \quad t \geq 0.$$

Note that L is unbounded (as $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$). However, S_t are bounded for $t \geq 0$, and moreover the family of linear operators $\{S_t \mid t \geq 0\}$ is a linear semigroup, since

$$S_0 = I \quad S_t \circ S_s = S_{t+s} \text{ for all } s, t \geq 0.$$

Roughly speaking, $S_t = e^{-Lt}$ (roughly because L is unbounded and the exponential formula is not necessarily valid).