Degree Theory & Some Aplications

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What does Degree Address?

Main question: Given f a continuous map from $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^n$ and $y \in \mathbb{R}^n$, does $f(x) = y$ have solutions?

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Three Axioms of a Degree

Defined formally, we want a function

$$d : \{(f, \Omega, y) : \Omega \subset \mathbb{R}^n \text{ open and bounded}, f : \overline{\Omega} \to \mathbb{R}^n \text{ continuous}, y \in \mathbb{R}^n \setminus f(\partial\Omega)\} \to \mathbb{Z}$$

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(d1) Normalization: *For identity,* $\mathrm{id}(x) = y \Leftrightarrow x = y$*, so*

$$d(\mathrm{id}, \Omega, y) = 1 \qquad \text{for all } y \in \Omega.$$

(d2) Additivity: *If* Ω_1, Ω_2 *partition* Ω*, then the number of solutions in* Ω_1, Ω_2 *should add up to that in* Ω:

$$d(f, \Omega_1, y) + d(f, \Omega_2, y) = d(f, \Omega, y)$$

if Ω_1, Ω_2 are open, $\Omega_1 \cap \Omega_2 = \emptyset$, and $y \notin f(\partial\Omega \setminus (\Omega_1 \cup \Omega_2))$.

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Three Axioms of a Degree (Cont.)

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satisfying the following three intuitive definitions:

(d3) Homotopy: $d(\cdot, \Omega, \cdot)$ *should remain fixed under continuous transformations (homotopy)*:

$$d(h(t, \cdot), \Omega, y(t)) \qquad \text{is independent of } t \in [0, 1]$$

if $h : [0, 1] \times \Omega \to \mathbb{R}^n$ and $y : [0, 1] \to \mathbb{R}^n$ are continuous and $y(t) \notin h(t, \partial\Omega)$ for all t. In particular, the following is useful:

$$d(h(0, \cdot), \Omega, y(0)) = d(h(1, \cdot), \Omega, y(1)).$$

A Remark

Claim. If $0 \notin h(\overline{\Omega})$ then $d(h, \Omega, 0) = 0$.

Proof. Since $d(h, \emptyset, 0) + d(h, \emptyset, 0) = 2d(h, \emptyset, 0) = d(h, \emptyset, 0)$ by additivity, $d(h, \emptyset, 0) = 0$. Since any open $\Omega_1 \subset \Omega$ satisfies $0 \notin h(\overline{\Omega} \setminus \Omega_1)$, by (d2) we have $d(h, \Omega_1, 0) = d(h, \Omega, 0) - d(h, \emptyset, 0) = d(h, \Omega, 0)$. In particular let $\Omega_1 := \emptyset$ so $d(h, \Omega, 0) = 0$ as well. □

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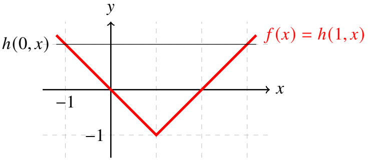
Example: Degree $\neq$ Number of Roots

Consider $f(x) = \begin{cases} -x & x \in [-1,1] \\ x-2 & x \in [1,3] \end{cases}$

and the homotopy



$$h(t,x) := 1 - t(2 - |x-1|)$$

so $h(1,x) = f(x)$ and $h(0,x)$ is the constant function 1. Also, $\partial(-1,3) = \{-1,3\}$ whereas $h(t,-1) = h(t,3) = 1 - t(2-2) = 1$ for all t. Hence setting $y(t) \equiv 1$ on $[0,1]$, we have

$$d(h(0,\cdot),(-1,3),y(0)) = d(h(1,\cdot),(-1,3),y(1)),$$

i.e.,

$$d(f,(-1,3),0) = \underbrace{d(\text{constant } 1,(-1,3),0)}_{=0} = 0.$$

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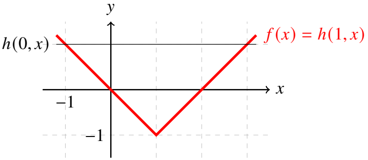

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Example: Degree Can Be Negative

A continuation: now consider $-\mathrm{id}$ on $(-1, 1)$ and a homotopy

$$h(t,x) := x - 2t \implies \begin{cases} h(0,\cdot) = \mathrm{id} \\ h(1,\cdot) = \mathrm{id} - 2 \end{cases} \quad \text{on } [-1,3].$$

Let $y(t) \equiv 0$ so $d(\mathrm{id} - 2, (-1,3), 0) = d(\mathrm{id}, (-1,3), 0) = 1$.
By the Remark above, $d(-\mathrm{id}, (1,3), 0) = 0$ so

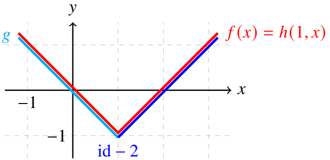
$$d(-\mathrm{id}, (-1,1), 0) = d(-\mathrm{id}, (-1,3), 0).$$

By the Remark again, $d(\mathrm{id} - 2, (-1,1), 0) = 0$ so by additivity

$$d(\mathrm{id} - 2, (1,3), 0) = 1.$$

Example: Degree Can Be Negative (Cont.)

Therefore,



$$0 = d(f, (-1,3), 0)$$
$$= d(f|_{(-1,1)}, (-1,1), 0) + d(f|_{(1,3)}, (1,3), 0)$$
$$= d(-\text{id}, (-1,1), 0) + \underbrace{d(\text{id} - 2, (1,3), 0)}_{=1}$$

so

$$d(-\text{id}, (-1,1), 0) = -1.$$

Uniqueness of Degree

Theorem: Existence and Uniqueness of Degree

There exists exactly one function $d(f, \Omega, y)$ satisfying the properties above. Furthermore, for linear functions represented by mulitplication by matrix A, the degree satisfies $d(A, \Omega, 0) = \operatorname{sgn}\det(A)$, whenever $\det(A) \neq 0$.

Proof is long and technical and therefore omitted.

In particular, the previous example $d(f, (-1, 3), 0) = 0$ does not violate this property as f is not linear.

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Theorem: Brouwer Fixed-Point Theorem (1911)

Let $f : \overline{B}_r(0) \to \overline{B}_r(0)$ be continuous where $B_r(0) \subset \mathbb{R}^n$. Then f admits a fixed point. More generally, $\overline{B}_r(0)$ can be replaced by any nonempty compact convex set in $\mathbb{R}^n$.

Proof. Consider the mapping $\mathrm{id} - f$. If f admits a fixed point on $\partial B_r(0)$ then we are done. Otherwise, let

$$h(t,x) := x - tf(x).$$

Then $h(1,\cdot) = \mathrm{id} - f$ and $h(0,\cdot) = \mathrm{id}$. By triangle inequality, for all $(t,x) \in [0,1) \times \partial B_r(0)$,

$$|h(t,x)| = |x - tf(x)| \geqslant |x| - t|f(x)| = r - t|f(x)| \geqslant (1-t)r \neq 0$$

and $f(x) \neq x$ on $\partial B_r(0) \implies |h(1,x)| \neq 0$ on $\partial B_r(0)$.

Let $y(t) \equiv 0$ on $[0,1]$. Hence g and f are homotopic and

$$d(\mathrm{id} - f, B_r(0), 0) = d(\mathrm{id}, B_r(0), 0) = 1,$$

so $(\mathrm{id} - f)(x) = 0$, i.e., $f(x) = x$, for some $x \in B_r(0)$. □

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Proof of the BFPT, General Case (skipped)

Proof of general case. Let D be compact convex and $f : D \to D$ continuous. There exists a continuous extension $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ defined as the following:

(1) Let $\{a_i\}$ be countable and dense in D.

(2) For each i and $x \notin A$, define $\varphi_i(x) = \max \left\{ 2 - \frac{|x - a_i|}{\text{dist}(x, D)}, 0 \right\}$, where $\text{dist}(x, D) := \inf\{|x - a| : a \in D\}$ is the distance.

(3) Define

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in D \\ \displaystyle\sum_{i=1}^{\infty} \frac{2^{-i}\varphi_i(x)}{\sum_{j=1}^{\infty} 2^{-j}\varphi_j(x)} f(a_i) & \text{if } x \notin D. \end{cases}$$

(*Take it for granted that this works!*)

Proof of the BFPT, General Case (skipped)

Proof (continued). For all x, $\tilde{f}(x)$ is the limit of convex combinations of $f(a_i) \in D$ ($\sum$ *coefficients* $= 1$ *and the claim follows by an* ϵ *argument on finite sums*), so $\tilde{f}(\mathbb{R}^n) \subset \overline{\operatorname{conv} f(D)} \subset D$.

Let r be such that $\overline{B}_r(0) \supset D$. Then $\tilde{f}(\overline{B}_r(0)) \subset D \subset \overline{B}_r(0)$. By the first case, $\tilde{f}(x) = x$ for some $x \in \overline{B}_r(0)$. But then $\tilde{f}(x) \subset D$ so $\tilde{f}(x) = f(x) = x$ for some $x \in D$. □

Application: Perron-Frobenius Theorem (if time permits)

Theorem: Perron-Frobenius Theorem

An $n \times n$ matrix A with $A_{i,j} \geqslant 0$ for all i, j has an eigenvalue $\lambda \geqslant 0$ corresponding to a *Perron-Frobenius eigenvector* $x \in \mathbb{R}^n$ whose components are all nonnegative.

Proof. Define $D := \{x \in \mathbb{R}^n : x_i \geqslant 0 \text{ and } \sum_{i=1}^n x_i = 1\}$. Note D is nonempty, compact, and convex.

If $Ax = 0$ for some $x \in D$ then the claim holds with $\lambda = 0$.

Otherwise, by compactness $\inf_{x \in D} \sum_{i=1}^n (Ax)_i \geqslant \alpha > 0$ for some α.

Then $x \mapsto Ax / \sum_{i=1}^n (Ax)_i$ defines a continuous map $f : D \to D$.

Brouwer's FPT guarantees a fixed point x, i.e.,

$$Ax = \left(\sum_{i=1}^n (Ax)_i\right) x =: \lambda x. \qquad \square$$

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The Borsuk-Ulam Theorem (1933)

Theorem: Borsuk-Ulam

Let $\Omega \subset \mathbb{R}^n$ be symmetric with $0 \in \Omega$. If $f : \partial\Omega \to \mathbb{R}^m$ is continuous and $m < n$, then $f(x) = f(-x)$ for some $x \in \partial\Omega$.

Famous results. Let $(m, n) = (2, 3)$. At any moment, there exists a pair of antipodal points with exactly the same temperature and pressure (assuming the parameters vary continuously in space).

Similarly, if $(m, n) = (1, 2)$, we see that, at any moment, there exists a pair of anti-equatorial points with the same temperature.

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Borsuk-Ulam: Proof

Proof. (*Red* = *"black box" because of limited time.*) Suppose not, then $g(x) := f(x) - f(-x) \neq 0$ on $\partial\Omega$.

We extend g on $\overline{\Omega}$ and embed its range to be $g(\overline{\Omega}) \times \{0\}^{n-m} \subset \mathbb{R}^n$. Call this function $\tilde{g}$. Then $0 \neq d(\tilde{g}, \Omega, 0) = d(\tilde{g}, \Omega, y)$ for all y sufficiently close to 0, say $y \in B_r(0)$.

But then for all $y \in B_r(0)$, $d(\tilde{g}, \Omega, 0) \neq 0$ so $y \in \tilde{g}(\overline{\Omega})$. Thus $B_r(0) \subset \tilde{g}(\overline{\Omega})$, but this is impossible due to the embedded $\{0\}^{n-m}$.

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Concluding Remarks

Claim. I will never reach this slide in this presentation.

Proof. Trivial. □