MATH 395: Problem Set on Calculus

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Problem 1

Prove that $e^x \geqslant 1 + x$ for all real x with equality if and only if $x = 0$.

Proof. Since the derivative of e^x and x are e^x and 1, respectively, and since $e^x > 1$ on $(0, \infty)$, by FTC we have

$$e^x = e^0 + \int_0^x e^t \,\mathrm{d}t = 1 + \int_0^x e^t \,\mathrm{d}t > 1 + \int_0^x 1 \,\mathrm{d}t = 1 + x \qquad \text{for all } x > 0.$$

Similarly, $e^x < 1$ on $(-\infty, 0)$, and a similar argument shows that $e^x > 1 + x$ on $(-\infty, 0)$ as well. Obviously $e^0 = 1 + 0$ so the claim follows. Problems 2 and 3 are analogous. □

Problem 4

Suppose $f : [0, 1] \to [0, 1]$ is continuous. Prove that f must have a fixed point.

Proof. Consider the continuous function $g(x) := f(x) - x$. By construction, $g(0) = f(0) \geqslant 0$ and $g(1) = f(1) - 1 \leqslant 0$. By the intermediate value property there must exist some $t \in [0, 1]$ with $g(t) = 0$, i.e., $f(t) = t$. □

Problem 5

Prove that there are infinitely many pairs of distinct positive real numbers x and y such that $x^x = y^y$.

Proof. If $x^x = y^y$, setting $t := x/y$ we have

$$(ty)^{ty} = t^{ty} y^{ty} = y^y \implies y = (t^{ty} y^{ty})^{1/y} = t^t y^t \implies y^{1-t} = t^t \implies \begin{cases} x = t^{1/(1-t)} \\ y = t^{t/(1-t)} \end{cases}.$$

In fact, this implication is $\Leftrightarrow$, i.e., for $t > 0$, $x := t^{1/(1-t)}$ and $y := t^{t/(1-t)}$ also satisfies $x^x = y^y$. Since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} t^{1/(1-t)} &= \frac{\mathrm{d}}{\mathrm{d}t} \exp(1/(1-t) \log t) \\ &= \exp(1/(1-t) \log t) \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\log t}{1-t} \right] \\ &= \exp(1/(1-t) \log t) \frac{(1-t) + t \log t}{t(1-t)^2} \\ &= \exp(1/(1-t) \log t) \frac{1 + t(\log t - 1)}{t(1-t)^2} > 0 \qquad \text{for } t > 0, \end{aligned}$$

$t^{t/(1-t)}$ is strictly increasing on $(0, \infty)$. Thus, there are uncountably many pairs of (x, y) such that $x^x = y^y$. □

Problem 6

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(0) = f(2)$. Prove that there exists c such that $f(c) = f(c-1)$.

Proof. Consider the continuous function $g(x) := f(x) - f(x-1)$. Then,

$$g(1) = f(1) - f(0) = -(f(2) - f(1)) = -g(2)$$

so there must exist some $t \in [1,2]$ with $g(t) = 0$, i.e., $f(t) = f(t-1)$. □

Problem 7

Compute $\int \sin^n x \, \mathrm{d}x$.

Solution Sketch. We can apply reduction of power as follows. First, using integration by parts,

$$\int \sin^n x \, \mathrm{d}x = \int \sin^{n-1} x \cdot \sin x \, \mathrm{d}x = -\sin^{n-1} x \cos x + (n-1) \int \cos^2 x \sin^{n-2} x \, \mathrm{d}x.$$

Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\int \sin^n x \, \mathrm{d}x = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, \mathrm{d}x - (n-1) \int \sin^n x \, \mathrm{d}x,$$

i.e.,

$$n \int \sin^n x \, \mathrm{d}x = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, \mathrm{d}x$$

or equivalently

$$\int \sin^n x \, \mathrm{d}x = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, \mathrm{d}x.$$ □

Problem 8

Compute $\int_{-1}^{1} \frac{\sqrt[3]{x}}{\sqrt[3]{1+x} + \sqrt[3]{1-x}} \, \mathrm{d}x$

Solution. Notice that this function is odd. The integral is therefore 0. □

Problem 9

Compute $\sum_{n=1}^{\infty} n 2^{-n}$ and $\sum_{n=1}^{\infty} n^2 2^{-n}$.

Solution. Since the series $\sum_{n=1}^{\infty} x^n$ has radius of convergence 1, the differentiated series $\sum_{n=1}^{\infty} n x^{n-1}$ also has the same radius of convergence by a result from analysis. That is,

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \qquad \sum_{n=1}^{\infty} n x^{n-1} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{1-x}\right) = \frac{1}{(1-x)^2} \qquad \text{for } |x| < 1.$$

Letting $x = 1/2$ we see $\sum_{n=1}^{\infty} n2^{-n+1} = 4$, so $\sum_{n=1}^{\infty} n2^{-n} = 2$.

Now consider the series $\sum_{n=1}^{\infty} nx^n$, which converges for $|x| < 1$ as the exponential growth rate $= |x| < 1$. Then, differentiating [*justification neeed?*] this series gives

$$\frac{1+x}{(1-x)^3} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{(1-x)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=1}^{\infty} nx^n = \sum_{n=1}^{\infty} n^2x^{n-1} \implies \sum_{n=1}^{\infty} n^2x^n = \frac{x(1+x)}{(1-x)^3}.$$

Letting $x := 1/2$ we have $\sum_{n=1}^{\infty} n^2x^n = 6$. □

Problem 10

Compute $\int_{-\infty}^{\infty} x^2e^{-x^2}\,\mathrm{d}x$.

Solution. Using (*hope this is not circular reasoning*?) the fact that a standard Gaussian Z has mean 0 and variance 1, we have the second moment $\mathbb{E}(X^2) = \mathrm{var}(X) + (\mathbb{E}X)^2 = 1$. Hence

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2e^{-x^2/2}\,\mathrm{d}x = 1 \implies \int_{-\infty}^{\infty} x^2e^{-x^2/2}\,\mathrm{d}x = \sqrt{2\pi}.$$

By u-substitution $u := x/\sqrt{2}$,

$$\int_{-\infty}^{\infty} 2\sqrt{2}u^2e^{-u^2}\,\mathrm{d}u = \int_{-\infty}^{\infty} x^2e^{-x^2/2}\,\mathrm{d}x \implies \int_{-\infty}^{\infty} u^2e^{-u^2}\,\mathrm{d}u = \int_{-\infty}^{\infty} x^2e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}/2.$$

Problem 11

Prove that for all $c \in \mathbb{R}$,

$$\int_0^{\pi/2} \frac{1}{1+\tan^c x}\,\mathrm{d}x = \pi/4.$$

Proof. We use the identities $\tan(a \pm b) = \dfrac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$ and $\tan(\pi/4) = 1$:

$$\begin{aligned}
\int_0^{\pi/2} \frac{1}{1+\tan^c x}\,\mathrm{d}x &= \int_0^{\pi/4} \frac{1}{1+\tan^c x}\,\mathrm{d}x + \int_{\pi/4}^{\pi/2} \frac{1}{1+\tan^c x}\,\mathrm{d}x \\
&= \int_0^{\pi/4} \frac{1}{1+\tan^c(\pi/4-x)}\,\mathrm{d}x + \int_0^{\pi/4} \frac{1}{1+\tan^c(\pi/2+x)}\,\mathrm{d}x \\
&= \int_0^{\pi/4} \frac{1}{1+(1-\tan x)^c/(1+\tan x)^c}\,\mathrm{d}x + \int_0^{\pi/4} \frac{1}{1+(1+\tan x)^c/(1-\tan x)^c}\,\mathrm{d}x \\
&= \int_0^{\pi/4} \frac{(1+\tan x)^c}{(1+\tan x)^c+(1-\tan x)^c}\,\mathrm{d}x = \int_0^{\pi/4} \frac{(1-\tan x)^c}{(1+\tan x)^c+(1-\tan x)^c}\,\mathrm{d}x \\
&= \int_0^{\pi/4} 1\,\mathrm{d}x = \pi/4.
\end{aligned}$$

□

Problem 12

Compute $\int_0^{\pi/4} \log(1+\tan x)\,\mathrm{d}x$.

Solution. Similar to the question above,

$$\begin{aligned}\int_0^{\pi/4} \log(1+\tan x)\,\mathrm{d}x &= \int_0^{\pi/4} \log(1+\tan(\pi/4-x))\,\mathrm{d}x \\ &= \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)\mathrm{d}x \\ &= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)\mathrm{d}x \\ &= \int_0^{\pi/4} \log 2 - \log(1+\tan x)\,\mathrm{d}x.\end{aligned}$$

Therefore,

$$\int_0^{\pi/4} \log(1+\tan x)\,\mathrm{d}x = \frac{1}{2}\int_0^{\pi/4} \log 2\,\mathrm{d}x = \frac{\pi\log 8}{2}.$$

Problem 13

Compute $\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x}\,\mathrm{d}x$.

Solution. Since $\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = \int_{-\pi}^{\pi} f(-t)\,\mathrm{d}t$, we have

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x}\,\mathrm{d}x &= \int_{-\pi}^{\pi} \frac{\sin(-nx)}{(1+2^{-x})\sin(-x)}\,\mathrm{d}x \\ &= \int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^{-x})\sin x}\,\mathrm{d}x = \int_{-\pi}^{\pi} \frac{2^x\sin(nx)}{(1+2^x)\sin x}\,\mathrm{d}x.\end{aligned}$$

Therefore,

$$2\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x}\,\mathrm{d}x = \int_{-\pi}^{\pi} \frac{(1+2^x)\sin(nx)}{(1+2^x)\sin x}\,\mathrm{d}x = \int_{-\pi}^{\pi} \frac{\sin(nx)}{\sin x}\,\mathrm{d}x.$$

Clearly, when $n=0$, the integral evaluates to $0/2=0$, and when $n=1$, the integral evaluates to $2\pi/2=\pi$. More generally, since

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{\sin((n+2)x)}{\sin x}\,\mathrm{d}x - \int_{-\pi}^{\pi} \frac{\sin(nx)}{\sin x}\,\mathrm{d}x &= \int_{-\pi}^{\pi} \frac{\sin((n+2)x)-\sin(nx)}{\sin x}\,\mathrm{d}x \\ &\overset{*}{=} \int_{-\pi}^{\pi} \frac{2\cos((n+1)x)\sin x}{\sin x}\,\mathrm{d}x \\ &= \int_{-\pi}^{\pi} 2\cos((n+1)x)\,\mathrm{d}x = 0\end{aligned}$$

(we used the sum-to-product formula $\sin a - \sin b = 2\cos((a+b)/2)\sin((a-b)/2)$ for the starred equation), the integral for general n is given by

$$\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x}\,\mathrm{d}x = \begin{cases} 0 & \text{if } n \text{ is even} \\ \pi & \text{if } n \text{ is odd} \end{cases}.$$

Problem 14

Simplify $1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \frac{x^{12}}{12!} + \cdots$.

Solution. By Taylor expansion,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{1}$$

$$e^{-x} = \sum_{k=0}^{\infty}(-1)^k \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots. \tag{2}$$

$$\cos x = \sum_{k=0}^{\infty}(-1)^{k+1} \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \tag{3}$$

It remains to notice that

$$\sum_{k=0}^{\infty} \frac{x^{4k}}{(4k)!} = \frac{1}{4}(1) + \frac{1}{4}(2) + \frac{1}{2}(3) = \frac{e^x + e^{-x}}{4} + \frac{\cos x}{2} = \frac{\cosh x + \cos x}{2}.$$

Problem 15

Simplify $\dfrac{1/1! + \pi^2/5! + \pi^8/9! + \pi^{12}/13! + \dots}{1/3! + \pi^4/7! + \pi^8/11! + \pi^{12}/15! + \dots}$.

Solution. Following the result of the previous problem, the numerator can be viewed as $(\cosh x + \cos x)/(2x)$ at $x = \pi$ (which accounts for the extra term in factorials), and the denominator can be viewed as $(\cosh x + \cos x)/(2x^3)$ at $x = \pi$. Their ratio is π^2.

Problem 16

Suppose f is a continuous function with $\int_0^1 f(x)\,\mathrm{d}x = 0$. Show there exists $t \in (0,1)$ such that

$$\int_0^t xf(x)\,\mathrm{d}x = 0.$$

Proof. Define $F(s) := \int_0^s f(x)\,\mathrm{d}x$ and $H(u) := \int_0^u F(s)\,\mathrm{d}s$.

Since f is continuous on $[0,1]$, we have $F \in C^1([0,1])$ and $H \in C^2([0,1])$. Writing F as H' and f as H'', integration by parts gives

$$\int_0^t xf(x)\,\mathrm{d}x = \int_0^t xH''(x)\,\mathrm{d}x = xH'(x)\Big|_{x=0}^{x=t} - \int_0^t H'(t)\,\mathrm{d}t = tH'(t) - H(t). \tag{1}$$

Since $H'(0) = F(0) = 0$ and $H'(1) = F(1) = \int_0^1 f(x)\,\mathrm{d}x = 0$ by assumption, there exists $t_0 \in (0,1)$ with

$$\frac{H(t_0) - H(0)}{t_0 - 0} = H'(t_0) \implies t_0H'(t_0) - H(t_0) \implies 0. \tag{*}$$

Therefore for such t_0 we have $\int_0^{t_0} xf(x)\,\mathrm{d}x = 0$ by (1).

Proof of (*). If $H(1) = 0$ then

$$1 \cdot H'(1) - H(1) = 0 - 0 = 0,$$

and we are done. Otherwise, this can be shown using Rolle's theorem. Note that $H(0) = 0$ by definition. Define

$$g(t) := \begin{cases} \dfrac{H(t)}{t} & \text{if } t \in (0,1] \\ 0 & \text{if } t = 0. \end{cases}$$

Since $\lim_{t \to 0+} H(t)/t = H'(0) = 0 = g(0)$, g is continuous at 0 and therefore on the entire $[0,1]$. It is clear that g is differentiable on $(0,1)$ so Rolle's theorem applies, so there exists $t_0 \in (0,1)$ such that

$$g'(t_0) = \frac{t_0 H'(t_0) - H(t_0)}{t^2} = 0 \implies t_0 H'(t_0) - H(t_0) = 0,$$

proving the claim. □

Problem 17

Suppose f, g are continuous real functions such that

$$g(x) = f(x) \int_0^x f(t)\, \mathrm{d}t.$$

Prove that if g is weakly decreasing then $f \equiv g \equiv 0$.

Proof. Suppose for contradiction that f is not identically 0. Thus, for some $x_0 \in \mathbb{R}$ we have $|f(x_0)| > 2c$ for some $c > 0$. WLOG assume $f(x_0) > 0$. By (ϵ, δ) continuity, there exists an open interval $(x_0 - \delta, x_0 + \delta)$ such that

$$f(x) \geqslant c \qquad \text{for all } x \in (x_0 - \delta, x_0 + \delta).$$

Then, for $x_0 - \delta < x_1 < x_2 < d_0 + \delta$, we have

$$\begin{aligned} g(x_2) - g(x_1) &= f(x_2) \int_0^{x_2} f(t)\, \mathrm{d}t - f(x_1) \int_0^{x_1} f(t)\, \mathrm{d}t \\ &\geqslant \min\{f(x_1), f(x_2)\} \left[\int_0^{x_2} f(t)\, \mathrm{d}t - \int_0^{x_1} f(t)\, \mathrm{d}t \right] \\ &\geqslant c \int_{x_1}^{x_2} f(t)\, \mathrm{d}t \geqslant (x_2 - x_1)c^2 > 0, \end{aligned}$$

contradicting the monotonicity of g. Hence f must be identically 0.
Now suppose for contradiction that g is not identically 0. This means $g(x_3) \neq 0$ for some $x_3 \in \mathbb{R}$, but then $f(x_3) \neq 0$, and our previous argument shows that this is a contradiction. Hence $f \equiv g \equiv 0$. □