MATH 395: Game Theory & Invariants

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Problem 1

Prove that a knight on a chessboard can only return to its original position after an even number of steps.

Proof. We color the chessboard in a checkered pattern using black and white. Then, after each move, the color corresponding to the knight's position alternates. Therefore it must take the knight an even number of steps to go to a position with the same color as the original position. In particular it will take an even number of steps for the knight to return to its original position. □

Problem 2

Prove that a generalized (p, q) knight move on an infinite chessboard can only return to its original position after an even number of steps.

Proof. WLOG assume p, q are co-prime, for otherwise we can shrink the chessboard by a factor of $\gcd(p, q)$. If we color the chessboard using a checkered pattern, then if $p+q$ is odd, the result follows from problem 1. Otherwise, if $p+q$ is even, we must have both p and q odd. Then, the horizontal shift in each step is odd, so it must take an even number of steps for the sum to add up to 0. (One can argue the same for vertical displacement, but based on how the problem is phrased, it suffices to only check the horizontal case.) □

Problem 3

Take an 8×8 chessboard and remove two diagonally opposite squares. Can the remaining board be tiled with 2×1 dominoes such that none overlap?

Solution. No. If we color the chessboard in a checkered pattern then there will be exactly 32 white and 32 black squares. The two removed must have the same color, so WLOG say there are 30 white and 32 black remaining. However, each domino must occupy exactly one white and one black square. Since $30 \neq 32$ it is impossible to tile this incomplete chessboard without overlap.

Problem 4

The game of double chess is the same as the game of chess except each player makes two moves in a row. Prove that the first player has a non-losing strategy.

Proof. Let A, B be the first and second player, respectively. We rephrase "A has a non-losing strategy" as

For all moves by B, there exists a strategy such that A does not lose

which can be negated as

(For all moves by the A,) there exists a strategy for the B to defeat A.

Assuming this negation is true, we let A move the knight and then move it back to the original position. Then, B confronts the exact same situation, as if nothing as happened, so it is reasonable to treat B as the actual "first player" and the A the actual "second player". Then, by the negated assumption, A has a winning strategy just like B, and this is clearly a contradiction. Thus A must have a non-losing strategy. ☐

Problem 5

You are given a list of numbers. You are allowed to replace two numbers a and b with $(a+b)/\sqrt{2}$ and $(a-b)/\sqrt{2}$. Can you transform $\{0,1,2,3\}$ into $\{\sqrt{2},\sqrt{2},2,2\}$ in finitely many steps?

Proof. No. Notice that

$$\left(\frac{a+b}{\sqrt{2}}\right)^2 + \left(\frac{a-b}{\sqrt{2}}\right)^2 = a^2 + b^2,$$

whereas

$$0^2 + 1^2 + 2^2 + 3^2 = 14 \neq 12 = (\sqrt{2})^2 + (\sqrt{2})^2 + 2^2 + 2^2.$$

Therefore the latter cannot be attained by such transformation on the former. ☐

Problem 6

You have a finite list of positive integers. You are allowed to pick two numbers and replace them with their gcd and lcm but only if this actually changes the numbers. Prove that you will eventually run out of allowed moves.

Proof. Note that $\{a,b\} = \{\gcd(a,b), \mathrm{lcm}(a,b)\}$ if and only if one divides the other, and when this is false,

$$\gcd(a,b) + \mathrm{lcm}(a,b) \geqslant 1 + 2a > a + b.$$

assuming WLOG $a > b$. Let $\{a_1^{(0)}, \dots, a_n^{(0)}\} \subset \mathbb{N}$ be the initial list and let $\{a_1^{(k)}, \dots, a_n^{(k)}\}$ be the list after k operations. Clearly, for all k we have

$$\sum_{i=1}^n a_i^{(k)} \leqslant \sum_{i=1}^n \mathrm{lcm}(a_1, \dots, a_n) \leqslant \sum_{i=1}^n \prod_{j=1}^n a_j = n \prod_{j=1}^n a_j < \infty.$$

Suppose we never run of of moves; that is, $\sum_{i=1}^n a_i^{(k)}$ is an strictly increasing sequence of integers (with k being the index). This contradicts the existence of an upper bound derived above, contradiction. Hence we must eventually run out of moves. □

Problem 7

You have 64 real numbers, 1 in each square of a chessboard. You are allowed to change the sign of all the numbers in a row or column. Can you reach an arrangement where all rows and columns have non-negative sum in finitely many steps?

Solution. Yes. Notation-wise, we let a 8×8 matrix $A^{(0)}$ to represent the original numbers on the chessboard and let $A^{(n)}$ to denote the numbers after n steps. We define the entry-wise sum

$$S_n := \sum_{i,j=1}^{n} A_{i,j}^{(n)}$$

Finally, we define a "negative row" to be one whose element-wise sum < 0(and likewise for "negative column").

Suppose there exists an initial layout $A^{(0)}$, we can never reach the desired state (i.e., no negative rows or columns). That is, at each step, there exists at least one negative row or column. To this end, we define the action at step n to be flipping the negative row/column in $A^{(n-1)}$. It is clear that under such action, $S_n > S_{n-1}$ as the corresponding row/column now becomes positive.
Since we began with 16 rows and columns to begin with, after altering the signs of each entry,there are only finitely many ways to obtain a strictly negative sum. The supremum over such combinations must therefore also be negative, i.e.,

$$\sup \sum_{i=1}^{8} x_i < 0 \qquad \text{subject to} \begin{cases} [x_1 \;\; x_2 \;\; \cdots \;\; x_8] \text{ is a row of } A^{(0)} \text{ or } (A^{(0)})^T \\ x_1 + \ldots + x_8 < 0. \end{cases} \tag{Δ}$$

The supremum is clearly attained by, say, $(y_1, \ldots, y_8)$, and furthermore, these also attain the following

$$\inf \left(\sum_{i=1}^{8} (-x_i) - \sum_{i=1}^{8} x_i \right) = -2 \inf \sum_{i=1}^{8} x_i \qquad \text{subject to the same conditions in } (\Delta).$$

Since $\liminf_{n \to \infty} S_n - S_{n-1} \geqslant -2 \inf \sum_{i=1}^{8} x_i > 0$, we see $\lim_{n \to \infty} S_n = \infty$, which is absurd, since

$$S_n \leqslant \sum_{i,j=1}^{n} |A_{i,j}^{(0)}| < \infty \qquad \text{for all } n.$$

Therefore we have a contradiction, so the desired state must be attained in finitely many steps, regardless of the initial state.

Remark. *Did you mean "64 integrs" instead? If so the claim would have been much simpler as no explicit justification of infimum is required. The proof would be no more than 4 lines in that case.*

Problem 8

You have 2014 stones in a pile. You are allowed to choose a pile, throw a stone away, and dividing the remaining stones into two smaller (nondegenerate) piles. Can you reach an arrangement where all your piles have only 1 stone?

Solution. No. Let the invariant be the number of stones left plus the number of piles we have, as each time we throw away a stone, we gain one pile. This invariant has value $2014 + 1 = 2015$ as implied by the initial state. If we manage to get one stone for one pile and there are k piles, then $2k = 2015$, a contradiction.

Problem 9

You have a stone at the origin in $\mathbb{Z}^2$. You are allowed to pick a stone with two unoccupied adjacent positions and replace the stone with two new stones at these adjacent positions. Prove that no matter what moves you make, there will always be a stone within distance 5 of the origin.

Solution. **I was given a two-word hint: "consider $2^{-|X|-|Y|}$".** If $\{(a_i, b_i)\}_{i=1}^n$ are the coordinates of n stones on $\mathbb{Z}^2$, we consider the "total weighted sum"

$$S := \sum_{i=1}^{n} 2^{-|a_i|-|b_i|}.$$

From the rule of the game, if we were to replace (x, y) by two adjacent coordinates (x_1, y_1) and (x_2, y_2), then

$$2^{-|x_1|+|y_1|} + 2^{-|x_2|+|y_2|} \geqslant 2^{-|x+1|-|y|} + 2^{-|x|-|y+1|} = 2^{-|x|-|y|}.$$

In other words, according to the game rule, the weight total sum will never be small than that of the initial state, i.e., $2^{-0-0} = 1$.

Suppose now for contradiction that at some state, all stones are of distance $\geqslant 6$ from the origin. Since the Euclidean 1-norm (i.e., $\|(x, y)\|_1 := |x| + |y|$) is bounded above by the standard Euclidean norm (i.e., $\|(x, y)\| := \sqrt{|x|^2 + |y|^2}$), the collection of all points with 1-norm $\leqslant 5$ is contained in the collection of all points with standard norm $\leqslant 5$. Thus the complement has the other direction of inclusion, and

$$\sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ \|\cdot\|\geqslant 6}} 2^{-|x|-|y|} \leqslant \sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ \|\cdot\|_1\geqslant 6}} 2^{-|x|-|y|} = \sum_{k=6}^{\infty} \sum_{|x|+|y|=k} 2^{-k}$$
$$= \sum_{k=6}^{\infty} \frac{4k}{2^{-k}} = 4 \sum_{k=6}^{\infty} \frac{k}{2^k} = \frac{4 \cdot 7}{2^5} = \frac{7}{8} < 1$$

(for the infinite sum step, see the calculus problem set's problem 9), strictly smaller than the total weight of our initial state. Contradiction.

Problem 10

There are 21 chips on a table. Alice and Bob play a game. On each term a player can remove $1, 2$, or 3 chips from the table. The player who removes the last chip wins. Alice plays first. Who wins?

Solution. Alice. She can pick one chip at first. Then if Bob picks x chips, she can pick $4 - x$ chips. After $1 + 2 \cdot 5$ turns, there will be no chip left, i.e., Alice will take the last chip, thereby winning the game.

Problem 11

In the same game with 21 chips, the rules are changed so whoever removes the last chip loses. Who wins now?

Solution. Bob. Each time Alice removes x chips, Bob simply needs to remove $4 - x$ so that there will be precisely 1 chip left after 5 rounds. Alice will have to pick it and lose the game.

Problem 12

There are 100 chips. Alice moves first, taking any number of chips between 1 and 99. In subsequent moves, each player must take at least 1 chip but no more than the previous move. The player who takes the last chip wins. Who wins?

Solution. Alice, starting by taking 4 chips.
Let us first note that, when the total number of chips left is even, whoever takes an odd number of chip loses, for then there will be an odd number of chips left, and the other person can simply take 1 chip, forcing both players to take one chip each turn from now on. By doing so the other player will take the last chip.
Therefore, with 96 chips left, Bob will have to take 2 or 4 chips. Alice's winning strategy is simply to take the same number of chips as Bob did in the previous move. Since 8 divides 96, if Bob always picks 4 chips, Alice will end up picking the last four and therefore the last chip. If Bob decides to pick 2 chips at a step where there are $96 - 8k$ chips remaining, then by the reasoning above they will each pick 2 chips from now on. Since 4 divides $96 - 8k$, Alice will end up picking the last chip again.

Problem 13

You have 2015 positive integers $a_1, \ldots, a_{2015}$. Let $b_1, \ldots, b_{2015}$ be an rearrangement. Prove that $(a_1 - b_1)\ldots(a_{2015} - b_{2015})$ is even.

Proof. It suffices to show that one among $(a_i - b_i)$, $1 \leqslant i \leqslant 2015$, is even. Suppose this is not true; then $\sum_{i=1}^{2015}(a_i - b_i)$, a sum of 2015 odd numbers, is odd. However, this is nothing but $\sum_{i=1}^{2015} a_i - \sum_{i=1}^{2015} b_i = 0$. Contradiction. □

Problem 14

A deck of cards is numbers from 1 to n in random order. Whenever the top card is k, reverse the order of the top k cards. Will you eventually get 1 on top?

Solution. Yes, and we prove by induction. The base case $n = 1$ is trivial.
For the inductive hypothesis, assume that for $k \in \mathbb{N}$, a deck of k cards will eventually get 1 on top. Now consider

a deck of $k+1$ cards:

(i) If the number $k+1$ is at the bottom of the deck, then it remain at the bottom throughout the game, as no number between 1 and k can retrieve $k+1$. Therefore the game reduces to a k-card deck, and we know we can get 1 on top by the inductive hypothesis.

(ii) If the number $k+1$ is not at the bottom:

- If 1 is at the bottom: we first consider the system consisting of $\{2, \ldots, k+1\}$, a k-card deck. We claim that the inductive hypothesis implies $k+1$ must show up on top after some time. (Otherwise replace the system $\{2, 3, \ldots, k, k+1\}$ by $\{2, 3, \ldots, k, 1\}$, and with the same move, 1 never gets on top, contradicting the hypothesis.) Then, the entire deck will be reversed and 1 will get on top.
- If 1 is not at the bottom: by the same argument, we note that the number of a card does not matter at all until it is retrieved and put on top. Let m be the bottom card. We can pretend as if $k+1$ is m until it is drawn. For now we have a k-card deck and we use the inductive hypothesis.
  - ⋆ If 1 appears on top before $k+1$ does, then we are done.
  - ⋆ If $k+1$ appears on top first, then it will swap with m after the next iteration. Then $k+1$ is now at bottom and the claim follows from (i).

Having done the induction, we claim 1 will always get on top,regardless of the size of deck.

Problem 15

Each room in a building has an even number of doors. Prove that the total number of exits is also even. (A door connects either two rooms to each other or one room to the outside and is called an exit in the latter case.)

Proof. Each door has two sides, so there will be a even number of "total sides," where each side can either face a room or the outside. Since each room has an even number of doors, summing over all rooms, the total number of sides facing toward rooms is even. Therefore, the total number of sides facing outward is even, and since there is a one-to-one correspondence between an outward-facing side and an exit, the number of exists is even. □

Problem 16

There are 1000 lilypads arranged in a circle and 1001 frogs distributed among those lilypads. Each minute, 2 frogs on the same lilypad jump to the adjacent lilypad in opposite directions. Prove that at some point there must be at least 501 occupied lilypads.

Proof. Pigeonhole implies that the jumping process will go on forever.

(Step 1). We show that each lilypad must be occupied at *some* point.

Proof. Suppose not, i.e., some lilypad is never visited by any frog. We number this lipypad as 0 and name the other lilypads as 1 to 999 counterclockwise. At time t, let $a(n,t)$ denote the lilypad the n^{th} frog sits on and we define

$$S_t := \sum_{n=1}^{1001} a(n,t)^2.$$

Since no frog can sit on lilypad 0, by construction no jump can take place on lilypads 1 or 999. However, for $2 \leqslant k \leqslant 998$, since $(k-1)^2 + (k+1)^2 = k^2 + 2$, in particular $S_{t+1} - S_t \geqslant 2$ for all t, as at least one jump takes place at each stage. (We define collective actions of two frogs sitting on the same lilypad as *one* jump.) However, each S_t is clearly bounded by $1001 \cdot 999^2 < \infty$ (as the largest lilypad number is 999). Contradiction. Therefore each lilypad must be occupied at *some* time.

(Step 2). By construction, once lilypad k becomes unoccupied, from then on, one between $\{k-1, k\}$ must have a frog, and one between $\{k, k+1\}$ must also have a frog (we do arithmetic in $\mathbb{Z}_{1000}$, i.e., $-1 = 999$ and $0 = 1000$). Partitioning the lilypads into $\{\{0,1\},\{2,3\},\ldots,\{998,999\}\}$ we see that after a sufficient amount of time (namely, use (Step 1) on all lilypads and take the max time), there will *always* be $\geqslant 500$ lilypads with frogs on them.

(Step 3) Let t_0 be sufficiently large so that at least one between lilypads $2k$ and $2k+1$ contains a frog at any given time $> t_0$. Suppose at time $t > t_0$ there are exactly 500 lilypads occupied. By pigeonhole there exists a lilypad x with 3 frogs on it. WLOG assume x is even with $x = 2k$.
Since lilypad x contains 3 frogs, at time $t+1$, lilypads x and $x+1$ will both become occupied. On the other hand, the remaining 998 lilypads must still have at least 499 occupied by assumption. Thus, at time $t+1$ we will have at least 501 occupied lilypads, proving the claim. □