

MATH 407 Problem Set 1

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From Notes on Set Theory:

(6) Prove that if $A \subset B \implies A - C \subset B - C$.

Proof. Pick any $x \in A - C$. It follows that $x \in A$ and $x \notin C$. Since $A \subset B$, $x \in A$ implies $x \in B$ and so $x \in B \wedge x \notin C$, i.e., $x \in B - C$. Hence $A - C \subset B - C$. \square

(7) Prove $B \subset C \implies A \times B \subset A \times C$.

Proof. Pick any $(x, y) \in A \times B$. It follows that $x \in A$ and $y \in B$. Since $B \subset C$, $y \in B$ implies $y \in C$ and so $x \in A \wedge y \in C$, i.e., $(x, y) \in A \times C$. Hence $A \times B \subset A \times C$. \square

(8) Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. “ \subset ”: pick any $x \in A \cup (B \cap C)$. Then either $x \in A$ or $x \in B \cap C$.

(I) If $x \in A$ then clearly $x \in A \cup B$ and $x \in A \cup C$, and so $x \in (A \cup B) \cap (A \cup C)$.

(II) If $x \in B \cap C$ then $x \in B$ and $x \in C$. Then $x \in B \implies x \in A \cup B$ and $x \in C \implies x \in A \cup C$ and so $x \in$ their intersection too, i.e., $x \in (A \cup B) \cap (A \cup C)$.

Hence “ \subset ” has been proven. Now we show “ \supset ”: pick any $y \in (A \cup B) \cap (A \cup C)$. By definition $y \in A \cup B$ and $y \in A \cup C$.

(I) If $y \in A$ then the claim holds.

(II) If $y \notin A$ then to make sure $y \in A \cup B$ we need $y \in B$, and likewise $y \in C$. Hence $y \in B \cap C$.

In either case, we see that $y \in (A \cup B) \cap (A \cup C)$ implies $y \in A \cup (B \cap C)$. Hence the LHS \subset the RHS, and we therefore conclude that indeed the two sides are equal. \square

(9) Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. “ \subset ”: pick any $x \in A \cap (B \cup C)$. By assumption we have $x \in A$ and $x \in B \cup C$, i.e., ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$). This means precisely $x \in (A \cap B) \cup (A \cap C)$.

“ \supset ”: now pick any $y \in (A \cap B) \cup (A \cap C)$. This means either $y \in A \cap B$ or $y \in A \cap C$. Immediately we can

confirm $y \in A$, and at least one between $(y \in B)$ and $(y \in C)$ is true, i.e., $y \in B \cup C$. Therefore $y \in A \cap (B \cup C)$.
Having shown both “ \subset ” and “ \supset ”, we conclude that the two sides are indeed equal. \square

(10) (De Morgan’s) Prove $(A \cap B)^c = A^c \cup B^c$.[†]

Proof. Since both directions follow a similar logic (simply apply definitions of \cup, \cap , etc.), from now on, instead of showing “ \subset ” and “ \supset ” separately I will show both together using a chain of \iff statements. Note that the chain of \implies ’s show \subset whereas the chain of \impliedby ’s show \supset .

$$\begin{aligned} x \in (A \cap B)^c &\iff x \notin A \cap B \\ &\iff x \notin A \text{ or } x \notin B \\ &\iff x \in A^c \text{ or } x \in B^c \\ &\iff x \in A^c \cup B^c. \end{aligned} \quad \square$$

(11) (De Morgan’s) Prove $(A \cup B)^c = A^c \cap B^c$.

Proof.

$$\begin{aligned} x \in (A \cup B)^c &\iff x \notin A \cup B \\ &\iff x \notin A \text{ and } x \notin B \\ &\iff x \in A^c \text{ and } x \in B^c \\ &\iff x \in A^c \cap B^c. \end{aligned} \quad \square$$

(12) Prove $A - (B \cap C) = (A - B) \cup (A - C)$.

Proof. (This entire statement is equivalent to (10) if A can be treated as a universal set.)

$$\begin{aligned} x \in A - (B \cap C) &\iff x \in A \text{ and } x \notin B \cap C \\ &\iff x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ &\iff (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\ &\iff x \in A - B \text{ or } x \in A - C \\ &\iff x \in (A - B) \cup (A - C). \end{aligned} \quad \square$$

(13) Prove $A - (B \cup C) = (A - B) \cap (A - C)$.

Proof.

$$\begin{aligned} x \in A - (B \cup C) &\iff x \in A \text{ and } x \notin B \cup C \\ &\iff x \in A \text{ and } (x \notin B \text{ and } x \notin C) \\ &\iff (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C) \\ &\iff x \in (A - B) \cap (A - C). \end{aligned} \quad \square$$

[†]Assuming U is the universal set, $\bar{A} := U - A = A^c$; even though \bar{A} doesn’t cause any confusion here, it may in the future (e.g. closure), so I prefer to use A^c to denote the complement.

(21) Prove $A \subset B$ if and only if $A - B = \emptyset$.

Proof.

$$A \subset B \iff (x \in A \implies x \in B)$$

$$\iff \{x : x \in A \text{ but } x \notin B\} = \emptyset$$

$$\iff A - B = \emptyset. \quad \square$$

(22) Prove $A \subset B$ if and only if $A \cap B = A$.

Proof. Notice that, $A \cap B = \{x : x \in A \wedge x \in B\}$ is always a subset of A , regardless of whether $A \subset B$ or not. It suffices to show ($A \subset B$ if and only if $A \subset A \cap B$), which can be proven as below:

$$A \subset B \iff (x \in A \implies x \in B)$$

$$\iff (x \in A \implies x \in A \text{ and } x \in B)$$

$$\iff (x \in A \implies x \in A \cap B)$$

$$\iff A \subset A \cap B. \quad \square$$