MATH 407 Problem Set 1

Qilin Ye

January 22, 2021

From `Notes on Set Theory`:

(6) Prove that if $A \subset B \implies A - C \subset B - C$.

Proof. Pick any $x \in A - C$. It follows that $x \in A$ and $x \notin C$. Since $A \subset B$, $x \in A$ implies $x \in B$ and so $x \in B \wedge x \notin C$, i.e., $x \in B - C$. Hence $A - C \subset B - C$. □

(7) Prove $B \subset C \implies A \times B \subset A \times C$.

Proof. Pick any $(x, y) \in A \times B$. It follows that $x \in A$ and $y \in B$. Since $B \subset C$, $y \in B$ implies $y \in C$ and so $x \in A \wedge y \in C$, i.e., $(x, y) \in A \times C$. Hence $A \times B \subset A \times C$. □

(8) Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. "$\subset$": pick any $x \in A \cup (B \cap C)$. Then either $x \in A$ or $x \in B \cap C$.

(I) If $x \in A$ then clearly $x \in A \cup B$ and $x \in A \cup C$, and so $x \in (A \cup B) \cap (A \cup C)$.

(II) If $x \in B \cap C$ then $x \in B$ and $x \in C$. Then $x \in B \implies x \in A \cup B$ and $x \in C \implies x \in A \cup C$ and so $x \in$ their intersection too, i.e., $x \in (A \cup B) \cap (A \cup C)$.

Hence "$\subset$" has been proven. Now we show "$\supset$": pick any $y \in (A \cup B) \cap (A \cup C)$. By definition $y \in A \cup B$ and $y \in A \cup C$.

(I) If $y \in A$ then the claim holds.

(II) If $y \notin A$ then to make sure $y \in A \cup B$ we need $y \in B$, and likewise $y \in C$. Hence $y \in B \cap C$.

In either case, we see that $y \in (A \cup B) \cap (A \cup C)$ implies $y \in A \cup (B \cap C)$. Hence the LHS $\subset$ the RHS, and we therefore conclude that indeed the two sides are equal. □

(9) Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. "$\subset$": pick any $x \in A \cap (B \cup C)$. By assumption we have $x \in A$ *and* $x \in B \cup C$, i.e., ($x \in A$ and $x \in B$) *or* ($x \in A$ and $x \in C$). This means precisely $x \in (A \cap B) \cup (A \cap C)$.
"$\supset$": now pick any $y \in (A \cap B) \cup (A \cap C)$. This means either $y \in A \cap B$ or $y \in A \cap C$. Immediately we can

confirm $y \in A$, *and* at least one between ($y \in B$) and ($y \in C$) is true, i.e., $y \in B \cup C$. Therefore $y \in A \cap (B \cup C)$. Having shown both "$\subset$" and "$\supset$", we conclude that the two sides are indeed equal. □

(10) (De Morgan's) Prove $(A \cap B)^c = A^c \cup B^c$.[†]

Proof. Since both directions follow a similar logic (simply apply definitions of $\cup, \cap$, etc.), from now on, instead of showing "$\subset$" and "$\supset$" separately I will show both together using a chain of $\iff$ statements. Note that the chain of $\implies$'s show $\subset$ whereas the chain of $\implies$'s show $\supset$.

$$\begin{aligned} x \in (A \cap B)^c &\iff x \notin A \cap B \\ &\iff x \notin A \text{ or } x \notin B \\ &\iff x \in A^c \text{ or } x \in B^c \\ &\iff x \in A^c \cup B^c. \end{aligned}$$ □

(11) (De Morgan's) Prove $(A \cup B)^c = A^c \cap B^c$.

Proof.
$$\begin{aligned} x \in (A \cup B)^c &\iff x \notin A \cup B \\ &\iff x \notin A \text{ and } x \notin B \\ &\iff x \in A^c \text{ and } x \in B^c \\ &\iff x \in A^c \cap B^c. \end{aligned}$$ □

(12) Prove $A - (B \cap C) = (A - B) \cup (A - C)$.

Proof. (This entire statement is equivalent to (10) if A can be treated as a universal set.)

$$\begin{aligned} x \in A - (B \cap C) &\iff x \in A \text{ and } x \notin B \cap C \\ &\iff x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ &\iff (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\ &\iff x \in A - B \text{ or } x \in A - C \\ &\iff x \in (A - B) \cup (A - C). \end{aligned}$$ □

(13) Prove $A - (B \cup C) = (A - B) \cap (A - C)$.

Proof.
$$\begin{aligned} x \in A - (B \cup C) &\iff x \in A \text{ and } x \notin B \cup C \\ &\iff x \in A \text{ and } (x \notin B \text{ and } x \notin C) \\ &\iff (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C) \\ &\iff x \in (A - B) \cap (A - C). \end{aligned}$$ □

[†]Assuming U is the universal set, $\overline{A} := U - A = A^c$; even though $\overline{A}$ doesn't cause any confusion here, it may in the future (e.g. closure), so I prefer to use A^c to denote the complement.

(21) Prove $A \subset B$ if and only if $A - B = \varnothing$.

Proof.

$$\begin{aligned} A \subset B &\iff (x \in A \implies x \in B) \\ &\iff \{x : x \in A \text{ but } x \notin B\} = \varnothing \\ &\iff A - B = \varnothing. \end{aligned}$$

□

(22) Prove $A \subset B$ if and only if $A \cap B = A$.

Proof. Notice that, $A \cap B = \{x : x \in A \wedge x \in B\}$ is always a subset of A, regardless of whether $A \subset B$ or not. It suffices to show ($A \subset B$ if and only if $A \subset A \cap B$), which can be proven as below:

$$\begin{aligned} A \subset B &\iff (x \in A \implies x \in B) \\ &\iff (x \in A \implies x \in A \text{ and } x \in B) \\ &\iff (x \in A \implies x \in A \cap B) \\ &\iff A \subset A \cap B. \end{aligned}$$

□