MATH 407 Homework 10

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Ex.5.39 To have two real roots, the discriminant needs to be nonnegative, i.e.,

$$(4Y)^2 - 16(Y+2) = 16(Y^2 - Y - 2) \geqslant 0 \implies 16(Y+1)(Y-2) \geqslant 0$$

and since Y is assumed to take values in $(0,5)$, we need $Y \geqslant 2$. Hence the probability is

$$P(Y \geqslant 2) = \frac{5-2}{5} = \frac{3}{5}.$$

Ex.5.42 $F_Y(y) = P(Y \leqslant y) = P(e^X \leqslant y) = P(X \leqslant \log(y)) = F_X(\log y) = \log y$. Then taking derivative gives

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} \log y = \frac{\chi_{(1,e)}(y)}{y}$$

TE 5.20 Recall from lecture that the MGF of a Gamma r.v. is $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha$. The mean is $M_X'(0) = \alpha/\lambda$. Hence

$$\begin{aligned}
\sigma_X^2 = \mathrm{Var}(X) = E[X^2] - \mu_X^2 &= M_X''(0) - \left(\frac{\alpha}{\lambda}\right)^2 \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\lambda^\alpha(\lambda - t)^{-\alpha}\right] - \left(\frac{\alpha}{\lambda}\right)^2 \text{ at } t = 0 \\
&= \lambda^\alpha \frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha(\lambda - t)^{-\alpha-1}\right) - \left(\frac{\alpha}{\lambda}\right)^2 \text{ at } t = 0 \\
&= \lambda^\alpha \alpha(\alpha+1)(\lambda - t)^{-\alpha-2} - \left(\frac{\alpha}{\lambda}\right)^2 \text{ at } t = 0 \\
&= \lambda^\alpha \alpha(\alpha+1)\lambda^{-\alpha-2} - \left(\frac{\alpha}{\lambda}\right)^2 \\
&= \frac{\alpha}{\lambda^2}.
\end{aligned}$$

TE 5.28 (1) We want to find the maximum of the pdf. Since the factor $1/B(\alpha, \beta)$ does not affect it we may simply disregard it. Then, differentiating the remaining terms w.r.t. x gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{a-1}(1-x)^{b-1}\right] &= x^{a-2}(1-x)^{b-2}\left[(a-1)(1-x) - (b-1)x\right] \\
&= x^{a-2}(1-x)^{b-2}\left[(a-1) - x(a+b-2)\right]
\end{aligned}$$

which = 0 if and only if $(a-1) = x(a+b-2)$. Indeed the solution is between $(0,1)$ and it is unique. Then since the pdf at $(a-1)/(a+b-2)$ is positive but $\to 0$ as $x \to 0$ or $x \to 1$, we conclude that the pdf is concave down $(a-1)/(a+b-2)$ is indeed the unique mode.

(2) The second derivative at the critical point $(a-1)/(a+b-2)$ (whether or not it's in $(0,1)$) is given by

$$f''\left(\frac{a-1}{a+b-2}\right) = \frac{(a-1)(1-b)}{a+b-2}\left(\frac{a-1}{a+b-2}\right)^{a-3}\left(\frac{b-1}{a+b-2}\right)^{b-3}.$$

This is positive so the graph is concave up, i.e., this critical point is a minimum not maximum. Therefore we need to check the pdf as $x \to 0$ or $x \to 1$, both of which blows up to ∞. Hence the mode can either be one of them or both, but it cannot be anything else.

(3) If $a = b = 1$ we have a uniform distribution and clearly the pdf is just constant and every $x \in (0,1) \cup \{0,1\}$ is a mode.

TE 5.29 For $x \in [0,1]$, $P(Y \leqslant x) = P(F(X) \leqslant x) = P(X \leqslant F^{-1}(x)) = FF^{-1}(x) = x$ and so the pdf of Y is simply uniformly 1, i.e., Y is a uniform distribution over $(0,1)$.

TE 5.31

$$F_Y(x) = F(Y \leqslant x) = F(X \leqslant \log X) = F_X(\log x)$$

Therefore,

$$f_Y(x) = \frac{\mathrm{d}}{\mathrm{d}x}(F_X(\log x)) = \frac{f_X(\log x)}{x} = \frac{1}{x\sigma\sqrt{2\pi}}\exp\left(-\frac{(\log x-\mu)^2}{2\sigma^2}\right).$$