MATH 407 Homework 11

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Ex.6.1 (a) Let $D_1, D_2 \sim \mathrm{DU}(0,6,n)$ (i.e., the probability of $d = 1,\dots,6$ is uniformly $1/6$) so that the represent the rolling of the two dice. Then $X = \max(D_1, D_2)$ and $Y = D_1 + D_2$. Since $\min(R(D_1)) = \min(R(D_2)) = 1 > 0$ we immediately see that $D_1 + D_2 = \max(D_1,D_2) + \min(D_1,D_2)$ so $\underline{Y > X}$ always holds. On the other hand, since $Y = D_1 + D_2 \leqslant 2\max(D_1,D_2) = 2X$ we see that $\underline{Y \leqslant 2X}$.

Going further, notice that $Y = 2X$ if and only if the two rolls are identical, in which case once that number is fixed, the probability is $(1/6)^2 = 1/36$.

For the more general case where $Y < 2X$, if $\max(D_1,D_2) = m$ and $D_1 + D_2 = n > m$, then either $(D_1,D_2) = (m, n-m)$ or $(D_1,D_2) = (n-m, m)$. Hence this probability is $2/36$. To out formally,

$$P(X = m, Y = n) = \begin{cases} 1/18 & n \in (m, 2m) \\ 1/36 & n = 2m \\ 0 & \text{otherwise} \end{cases} \qquad \text{where} \qquad (m,n) \in \{1,\dots,6\} \times \{2,\dots,12\}.$$

(b) Let $X, D_2 \sim \mathrm{DU}(0,6,n)$. Immediately we see that $Y \geqslant X$. In fact we can directly compute the joint pmf:

$$P(X = m, Y = n) = \begin{cases} m/36 & m = n \ (Y \text{ can take any value}) \\ 1/36 & m < n \ (Y \text{ fixed too}) \\ 0 & \text{otherwise} \end{cases} \qquad \text{where } (m,n) \in \{1,\dots,6\} \times \{1,\dots,6\}.$$

(c) Let $D_1, D_2 \sim \mathrm{DU}(0,6,n)$. We have $X = \min(D_1,D_2)$ and $Y = \max(D_1,D_2)$. Immediately we see $Y \geqslant X$ and the case $Y = X$ is obtained if and only if $D_1 = D_2$. If $Y = y > x = X$ then again we have two possibilities: $(D_1,D_2) = (x,y)$ or (y,x). Hence

$$P(X = m, Y = n) = \begin{cases} 1/18 & m < n \\ 1/36 & m = n \\ 0 & \text{otherwise} \end{cases} \qquad \text{where } (m,n) \in \{1,\dots,6\} \times \{1,\dots,6\}.$$

Ex.6.8 (a) To find c we want the integral of the joint pdf to be 1, namely

$$\begin{aligned} \int_0^\infty \int_{-y}^y c(y^2 - x^2)e^{-y} \,\mathrm{d}x \,\mathrm{d}y &= c\int_0^\infty \int_{-y}^y y^2 e^{-y} - x^2 e^{-y} \,\mathrm{d}x \,\mathrm{d}y \\ &= c\int_0^\infty 2y^3 e^{-y} - \left[\frac{x^3 e^{-y}}{3}\right]_{x=-y}^{y} \mathrm{d}y \\ &= c\int_0^\infty 2y^3 e^{-y} - \frac{2y^3 e^{-y}}{3} \,\mathrm{d}y \\ &= \frac{4c}{3}\int_0^\infty y^3 e^{-y} \,\mathrm{d}y = \frac{4c\Gamma(4)}{3} = 8c = 1 \end{aligned}$$

so it follows that $c = 1/8$.

(b) We simply plug in the formulae:

$$f_Y(y) = \int_{-y}^{y} c(y^2 - x^2)e^{-y}\,\mathrm{d}x = \frac{y^3 e^{-y}}{6} \text{ for } y \in (0, \infty),$$

and (since given x, y is defined on $[|x|, \infty)$)

$$\begin{aligned} f_X(x) &= \int_0^\infty c(y^2 - x^2)e^{-y}\,\mathrm{d}y \\ &= c\int_{|x|}^\infty y^2 e^{-y} - x^2 e^{-y}\,\mathrm{d}y \\ &= \frac{1}{8}\left[-y^2 e^{-y} - 2ye^{-y} - 2e^{-y} + x^2 e^{-y}\right]_{y=|x|}^\infty \\ &= \frac{1}{8}\left(e^{-|x|}(2|x| + 2)\right) = \frac{e^{-|x|}(1 + |x|)}{4}. \end{aligned}$$

(c) Notice that

$$x f_X(x) = \frac{xe^{-|x|}(1 + |x|)}{4} = -\frac{(-x)e^{-|-x|}(1 + |-x|)}{4} = -(-x)f_X(-x),$$

so integrating over this odd function would evaluate to 0, given that the upper and limits are symmetric across 0.

Ex.6.9 (a) It is clear enough that as $x, y > 0$ the function $f(x, y) > 0$, so it suffices to check that the integral evaluates to 1:

$$\begin{aligned} \int_0^1 \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)\mathrm{d}y\,\mathrm{d}x &= \frac{6}{7}\int_0^1 \int_0^2 x^2 + xy/2\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{6}{7}\int_0^1 2x^2 + x\,\mathrm{d}x \\ &= \frac{6}{7}\left(\frac{2x^3}{3} + \frac{x^2}{2}\right)_{x=0}^1 \\ &= \frac{6}{7}\cdot\frac{7}{6} = 1. \end{aligned}$$

(b)

$$f_X(x) = \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)\mathrm{d}y = \frac{6}{7}\int_0^2 x^2 + xy/2\,\mathrm{d}y = \frac{6}{7}(2x^2 + x).$$

(c) $P(X > Y)$ is given by integrating the joint pdf over the region described by $\int_0^1 \int_0^x \mathrm{d}y\,\mathrm{d}x$:

$$\begin{aligned} P(X > Y) &= \int_0^1 \int_0^x \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)\mathrm{d}y\,\mathrm{d}x \\ &= \frac{6}{7}\int_0^1 x^3 + \frac{x^3}{4}\,\mathrm{d}x \\ &= \frac{6}{7}\left(\frac{1}{4} + \frac{1}{16}\right) = \frac{6}{7}\cdot\frac{5}{16} = \frac{15}{56}. \end{aligned}$$

(d) $P(Y > 1/2 \mid X < 1/2)$ is given by $\dfrac{P(Y > 1/2, X < 1/2)}{P(X < 1/2)}$ so we will need to compute both sides separately.

$$\begin{aligned} P(Y > 1/2, X < 1/2) &= \int_0^{1/2} \int_{1/2}^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)\mathrm{d}y\,\mathrm{d}x \\ &= \frac{6}{7}\int_0^{1/2} \frac{12x^2 + 9x}{8}\,\mathrm{d}x = \frac{69}{448} \end{aligned}$$

and

$$P(X < 1/2) = \int_0^{1/2} \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \, \mathrm{d}y \, \mathrm{d}x = \frac{6}{7}\int_0^{1/2} 2x^2 + x \, \mathrm{d}x = \frac{6}{7} \cdot \frac{5}{24} = \frac{5}{28}.$$

Therefore $P(Y > 1/2 \mid X < 1/2) = 69 \cdot 28/(448 \cdot 5) = 69/80$.

(e)

$$E[X] = \int_0^1 x f_X(x) \, \mathrm{d}x = \int_0^1 \frac{6}{7}(2x^3 + x^2) \, \mathrm{d}x = \frac{6}{7} \cdot \frac{5}{6} = \frac{5}{7}.$$

(f)

$$E[Y] = \int_0^2 y f_Y(y) \, \mathrm{d}y = \int_0^2 y \int_0^1 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \, \mathrm{d}x \, \mathrm{d}y$$
$$= \frac{6}{7}\int_0^2 y \cdot \left(\frac{1}{3} + \frac{y}{4}\right) \, \mathrm{d}y = \frac{6}{7}\int_0^2 \frac{y}{3} + \frac{y^2}{4} \, \mathrm{d}y = \frac{6}{7} \cdot \frac{4}{3} = \frac{8}{7}.$$

Ex.6.19 (*) Clearly $1/x$ is always positive for $0 < x < 1$ so it suffices to evaluate the integral:

$$\int_0^1 \int_0^x \frac{1}{x} \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 \, \mathrm{d}x = 1.$$

(a)

$$f_Y(y) = \int_y^1 \frac{1}{x} \, \mathrm{d}x = -\ln y \text{ for } y \in (0,1).$$

(b)

$$f_X(x) = \int_0^x \frac{1}{x} \, \mathrm{d}y = \frac{x}{x} = 1 \text{ for } x \in (0,1).$$

(c)

$$E[X] = \int_0^1 x \, \mathrm{d}x = \frac{1}{2}$$

(d)

$$E[Y] = \int_0^x -y \ln y \, \mathrm{d}y = \frac{1}{4}.$$

Ex.6.21 (a) Again, clearly given the conditions on x, y, $f(x,y) \geqslant 0$. It remains to check the integral:

$$\int_0^1 \int_0^{1-x} 24xy \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 12x(1-x)^2 \, \mathrm{d}x$$
$$= \int_0^1 12x - 24x^2 + 12x^3 \, \mathrm{d}x$$
$$= 6 - 8 + 3 = 1.$$

(b)

$$E[X] = \int_0^1 x f_X(x) \, \mathrm{d}x$$
$$= \int_0^1 x \int_0^{1-x} 24xy \, \mathrm{d}y \, \mathrm{d}x$$
$$= \int_0^1 12x^2(1-x)^2 \, \mathrm{d}x$$
$$= \int_0^1 12x^2 - 24x^3 + 12x^4 \, \mathrm{d}x = 4 - 6 + \frac{12}{5} = \frac{2}{5}.$$

(c) The conditions on x and y are completely symmetric; also $2/5$.